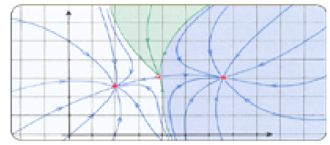
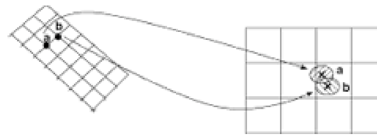
Part 1: Dynamics

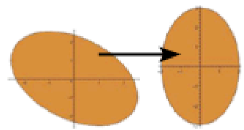
Apr 2 (01) **2-Dimensional flow geometries**. HW1



Apr 4 (02) **Discrete dynamics & Mappings**.

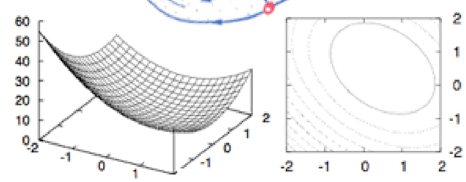


Apr 9 (03) **Diagonalization & eigenvalues**. HW2



Apr 11 (04) **Higher dimensional dynamics & linearization**.

Apr 16 (05) **Stability & Gradient systems**. HW3

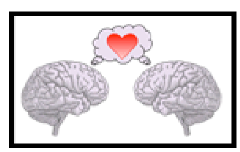


Dynamical Systems with Applications using MATLAB (2004) Stephen Lynch

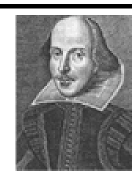
I. Dynamics: 2-Dimensional flow geometries

Steps to quantify a System's Dynamics:

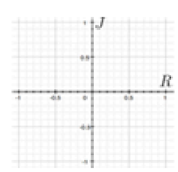
Step #1:*"Draw a distinction"*



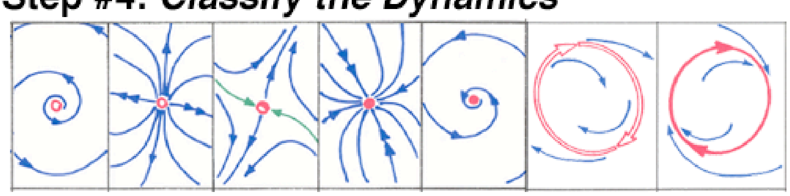
Step #2: *Specify an example*



Step #3: *Quantify the salient features*

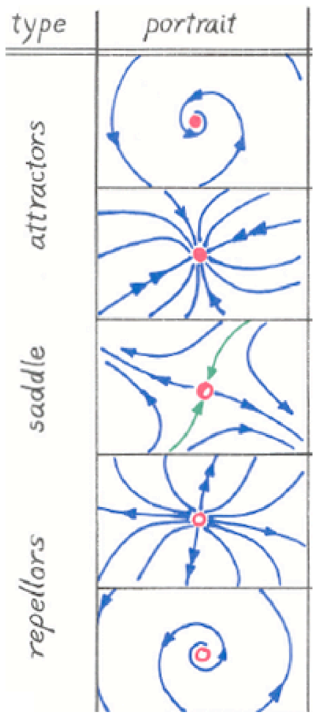


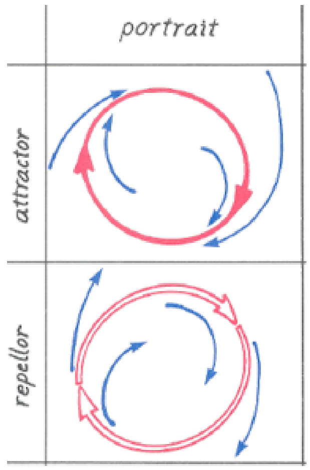
Step #4: *Classify the Dynamics*



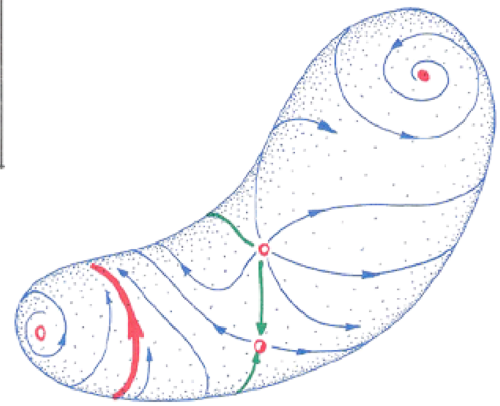
The Classification of 2-Dimensional Dynamics

Piexoto's Theorem

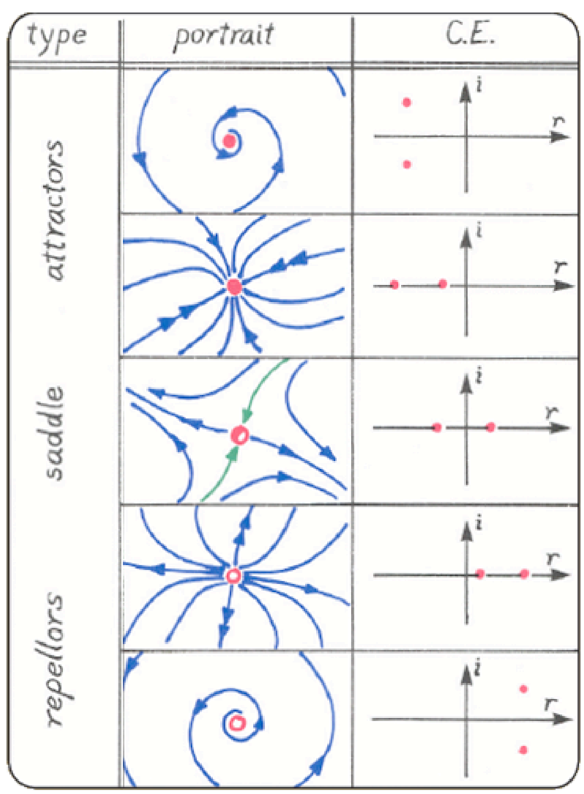




Compact surface



Two-Dimensions
Eigenvalues Determine the Geometry of Flows

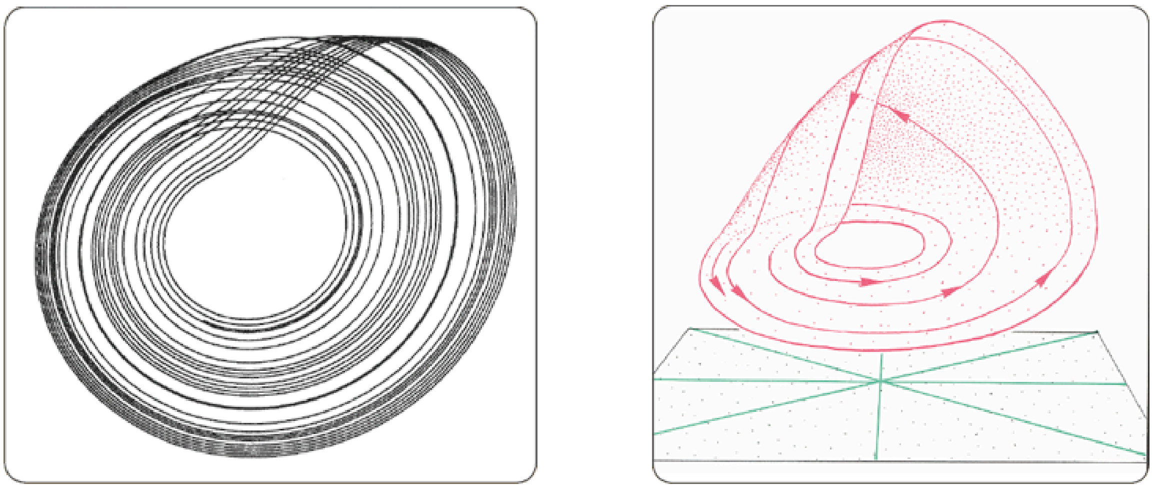


Add a Third Dimension

$$\frac{dx(t)}{dt} = -y(t) - z(t)$$

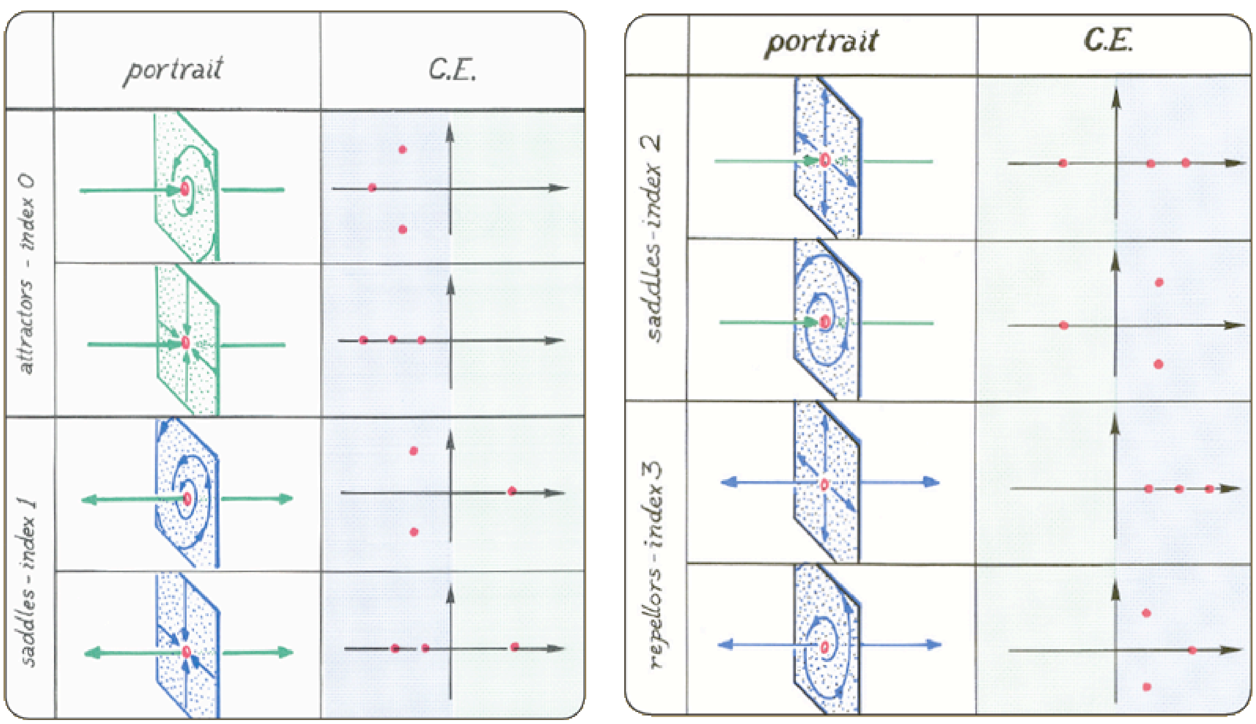
$$\frac{dy(t)}{dt} = x(t) + 0.15y(t)$$

$$\frac{dz(t)}{dt} = b + z(t)(x(t) - c)$$



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

In Any Dimension
Flow Geometry Follows from Eigenvalues



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

Lyapunov Exponent

The Lyapunov exponent is a measure of how much two neighboring initial points will diverge in the dynamics flow.

1-dimensional system: an initial separation, Δx_0.
The separation at a much later time will be given by

$$\Delta x_t = \Delta x_0 e^{\lambda t}$$

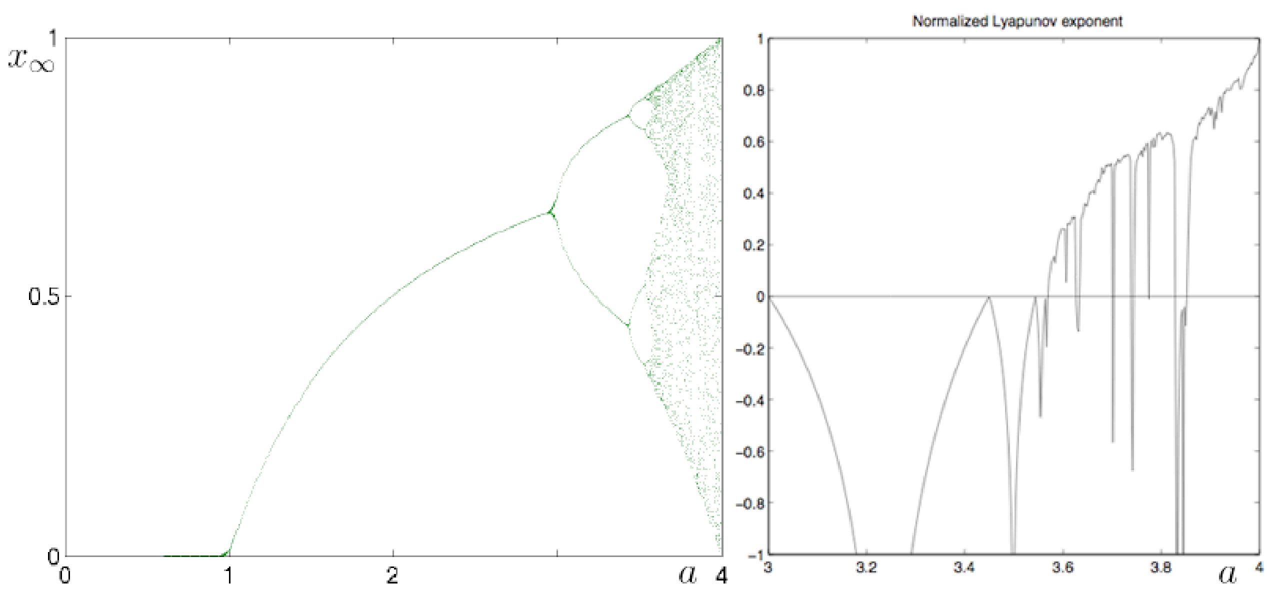
Where the Lyapunov exponent of the system is defined by

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln(\Delta x(t))$$

and $\Delta x(t)$ is the average deviation of the unperturbed trajectory.

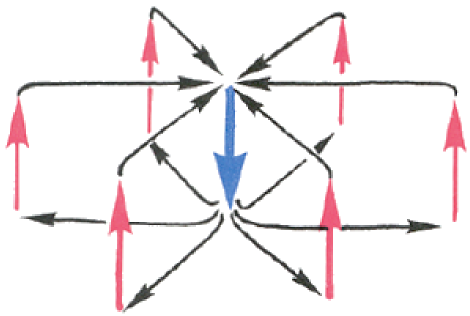
Lyapunov Exponent for the Logistic Map

$$\boxed{x_{n+1} = a x_n (1 - x_n)}$$



<http://courses.physics.kth.se/5A1352/>

Lorenz Equation (1963) Simplifies the Dynamics of a Single Cell



$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(R - z) - y\\ \dot{z} &= xy - bz\end{aligned}$$

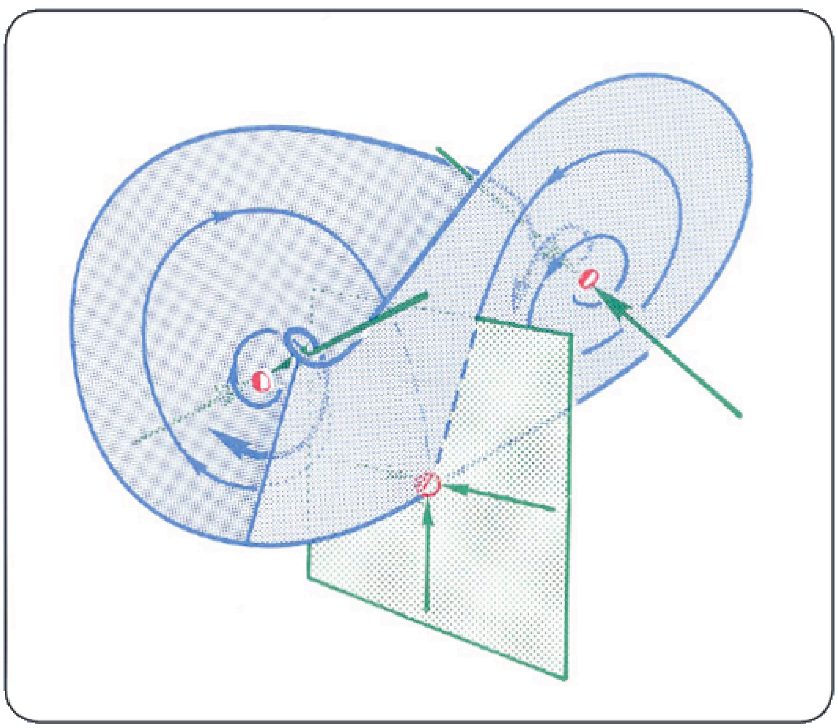
$\sigma = 10,\ R = 28,\ b = 8/3$

x = *rate of convection overturning*
y = *horozontal temperature gradient*
z = *vertical temperature gradient*

Ed Lorenz's computer simulations showed that the trajectories of solutions have a sensitive dependence on initial conditions.

The three parameter are positive and are the Prandtl number, the Rayleigh number, and a scaling factor.

Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

Lyapunov's Stability Theorem

To show that a system is stable, construct a **Lyapunov function**.

Lyapunov's stability theorem: If there is a Lyapunov function V such that:

$\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f(\bar{x}) = 0$, $\bar{x}$ is a fixed point.

$V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function defined on some neighborhood U of $\bar{x}$.

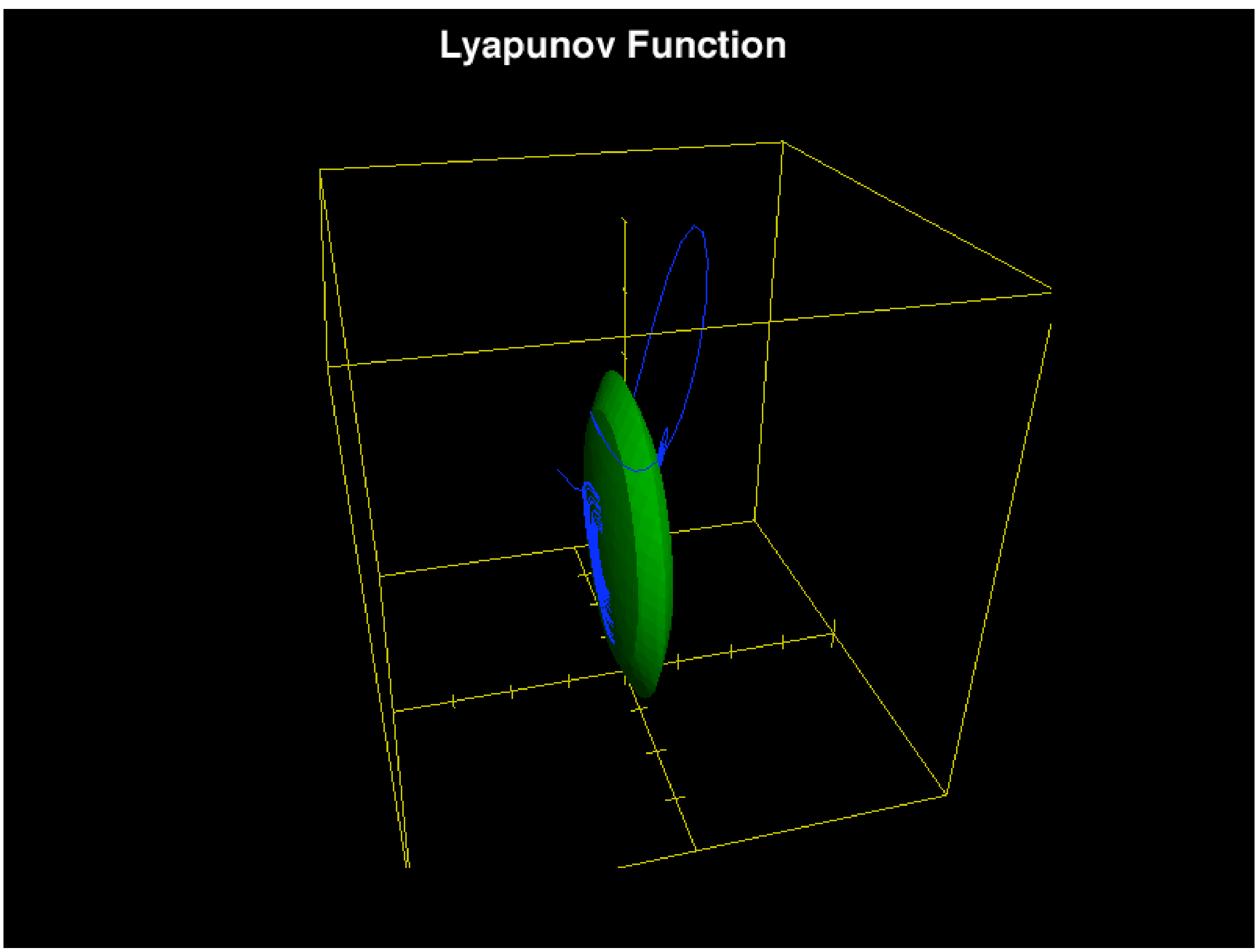
$V(\bar{x}) = 0$ and $V(x) > 0 \; \forall x \in (U - \bar{x})$.

$\dot{V} \leq 0 \; \forall x \in (U - \bar{x})$

then $\bar{x}$ is stable.

If $\dot{V} < 0 \; \forall x \in (U - \bar{x})$, then $\bar{x}$ is asymptotically stable.

(from Andy Fraser's notes)



Summary: Eigenvalues Determine Flow Geometries

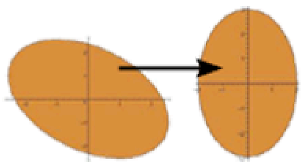
Classify the Dynamics: ***1. Find the fixed points.***

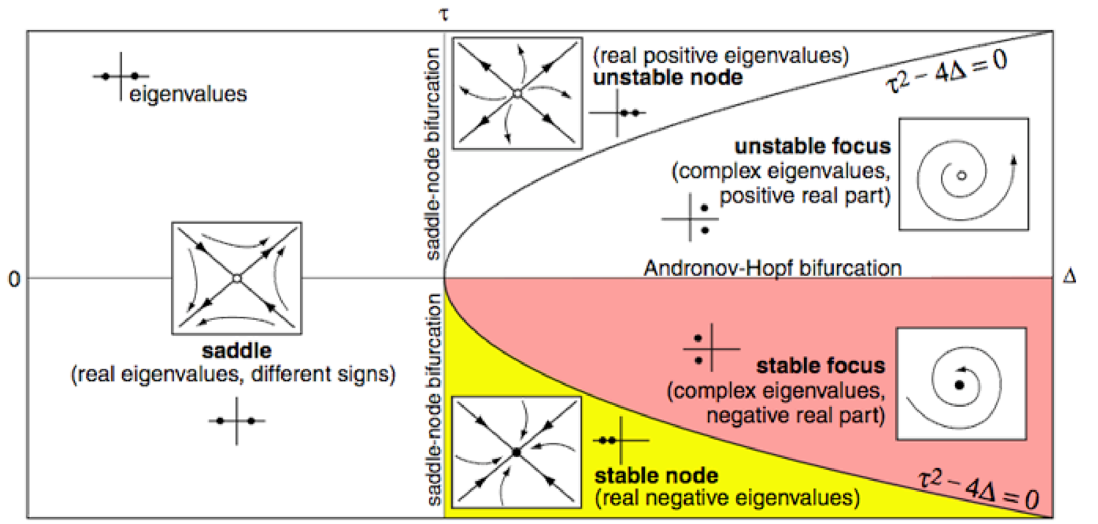
2. Linearize near the fixed points.

3. Compute eigenvalues at fixed points.

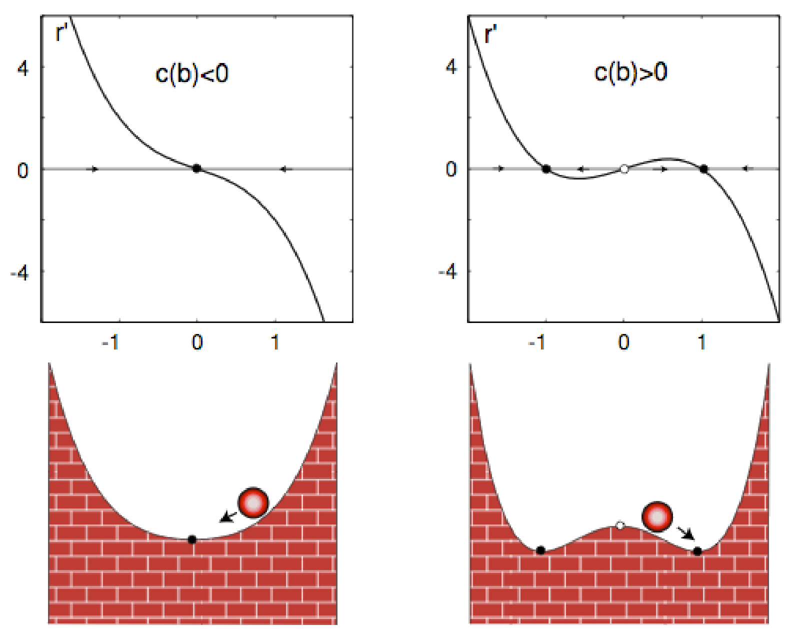
4. Classify local stability.

5. Classify bifurcations.



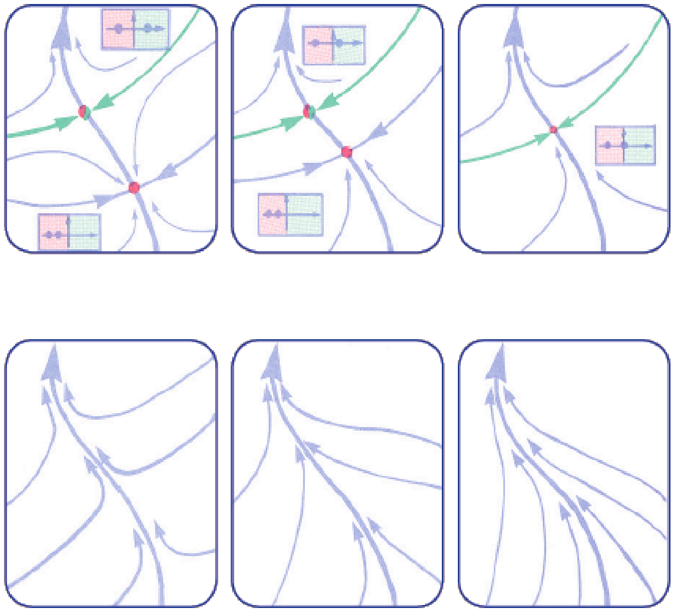


Bifurcations in 1-Dimensional Gradient Systems



Dynamical Systems in Neuroscience (2006) E.M. Izhikevich

Elementary Catastrophes: 2-D Fold
Saddle-Node Bifurcation



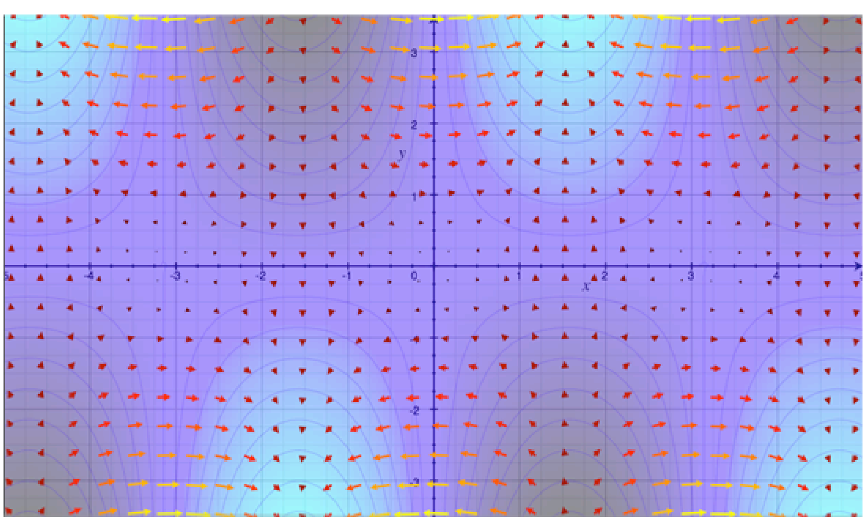
Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

Gradient Systems: Gradient of a Potential Functions

Vector fields associated to a scalar potential: $V : \mathbb{R}^n \longrightarrow \mathbb{R}$

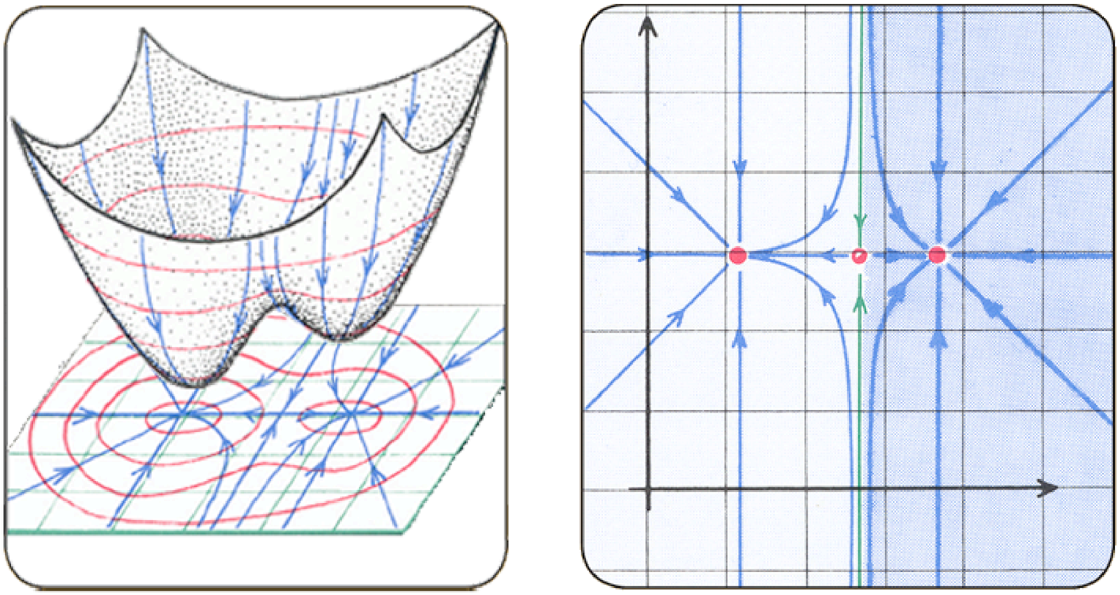
$$\frac{d}{dt}\vec{x} = \vec{f}(x) = -\nabla V(x)$$

The gradient is the maximal directional derivative: $\nabla V(x) \;=\; \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_N \end{bmatrix} V(x)$



$f(x,y)=y{\cdot}\sin x$

Gradient Systems: Forces from Potential Function



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

Gradient Systems: No Closed Orbits

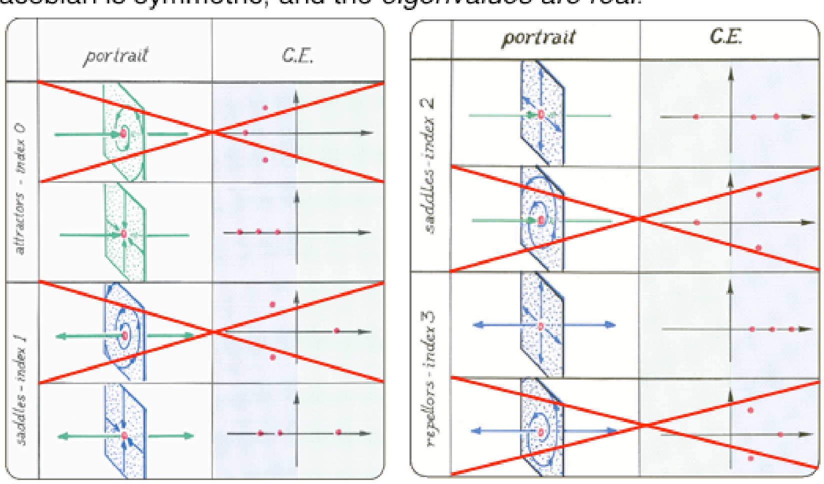
The *Jacobian* of the dynamical system,

$$\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}) = -\nabla V(\vec{x})$$

is the *Hessian* of the potential:

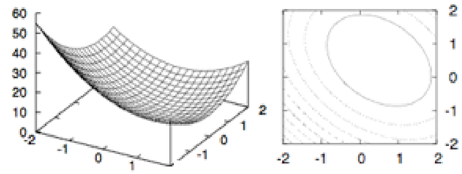
$$H_{i,j}(x) \equiv \frac{\partial^2}{\partial x_i \partial x_j} V(x)$$

Then the Jacobian is symmetric, and the *eigenvalues are real*.

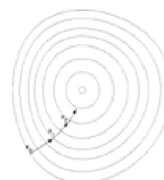


Part 2: Optimization

Apr 25 (07) **Optimization overview**.

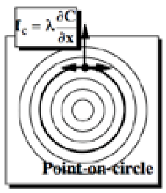


Apr 30 (08) **Dynamics of Optimization**.
(Practice midterm)

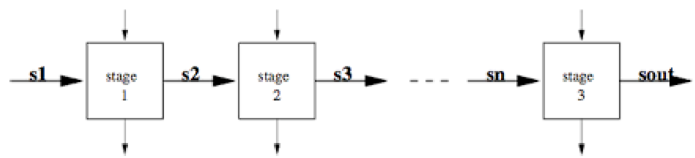


May 2 (09) Midterm.
May 7 (10) Review midterm.

May 9 (11) **Constrained optimization**. HW 5



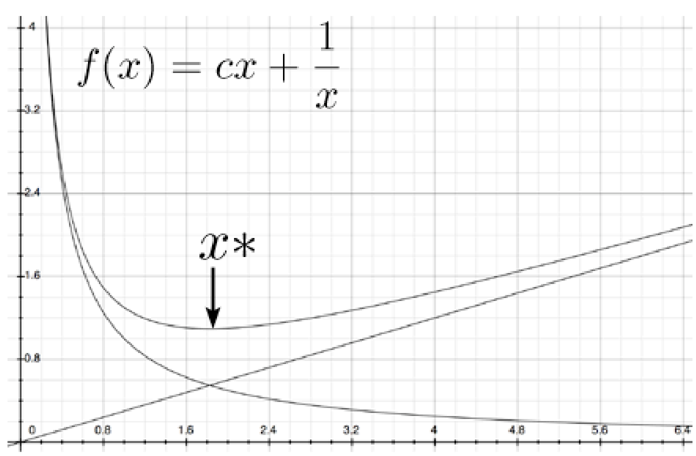
May 14(12) **Dynamic programming**.



Unconstrained Optimization

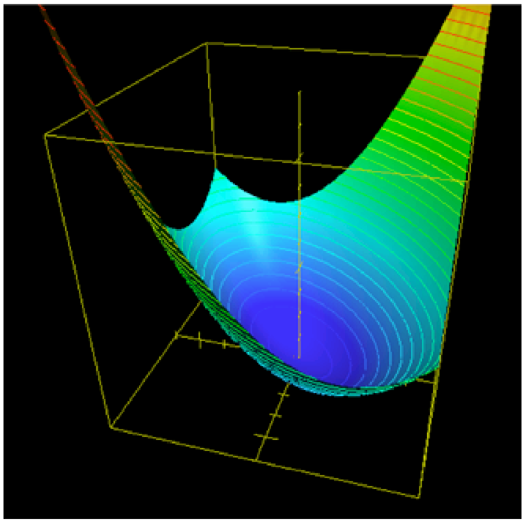
First order conditions: $$\frac{d}{dx} f(x)|_{x=x*} = 0$$

Second order conditions: $$\frac{d^2}{dx^2} f(x)|_{x=x*} > 0$$ (minimum)

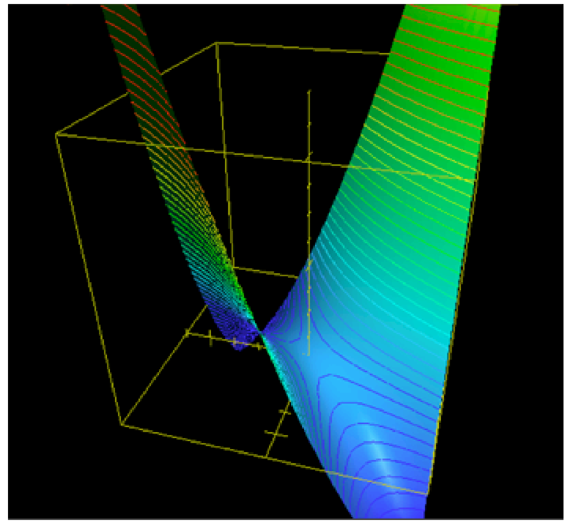


Unconstrained Optimization

$$f(x) = x_1^2 + x_2^2 + a x_1 x_2$$



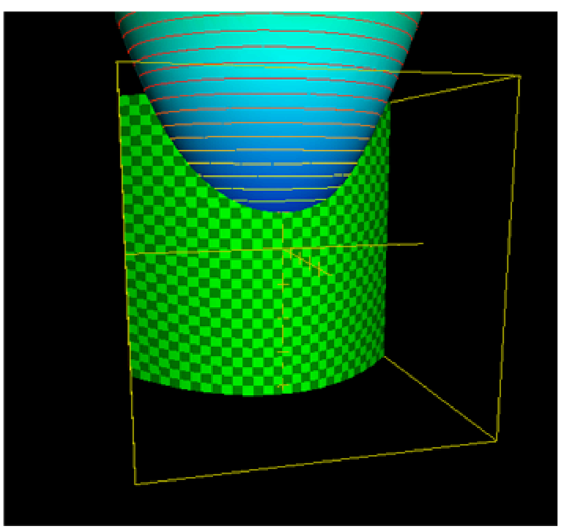
$a = 1$



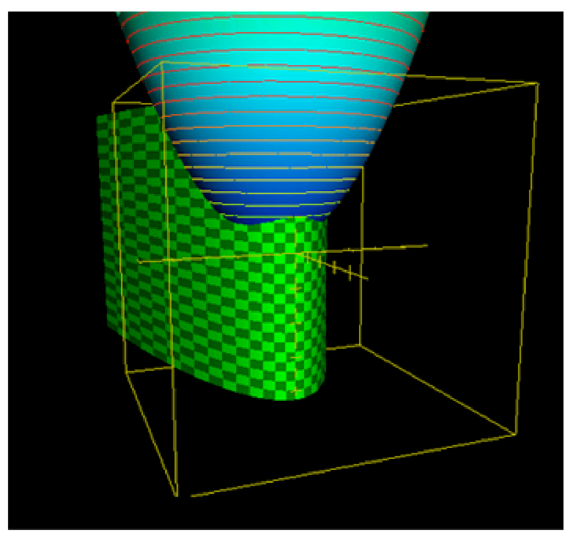
$a = 3$

Constrained Optimization

$$f(x) = 1/2((x_1 - 1)^2 + x_2^2)$$
$$c(x) = -x_1 + \beta x_2^2 = 0$$



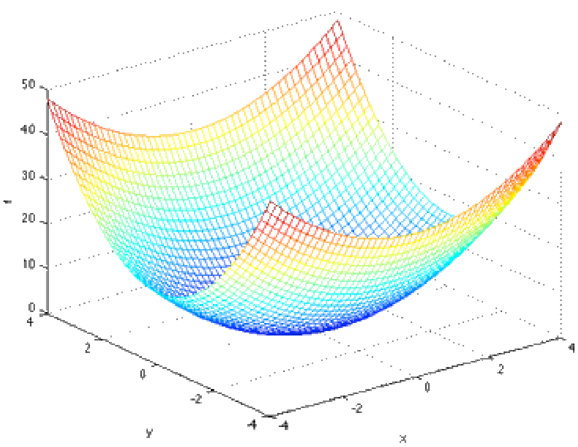
$\beta = 1/4$

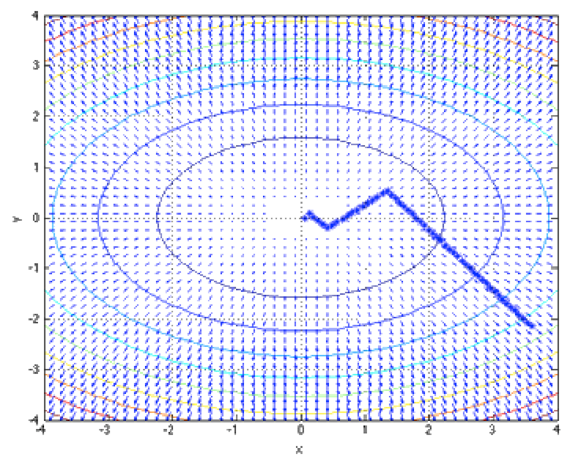


$\beta = 1$

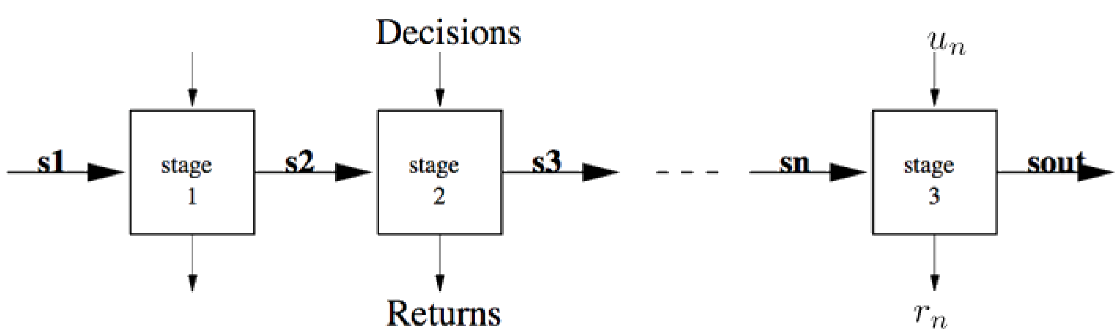
Gradient Descent

1. Choose random start point: $\vec{x}_0$
2. Compute the gradient: $\Delta f(\vec{x})|_{\vec{x}=\vec{x}_0}$
3. Compute the search direction: $\vec{s}_0 = \dfrac{-\Delta f(\vec{x}_0)}{||\Delta f(\vec{x}_0)||}$
4. Minimize the 1-dim function: $F(t) = f(\vec{x}_0 + t\vec{s}_0)$
5. Choose next point: $\vec{x}_1 = \vec{x}_0 + t * \vec{s}_0$
6. Iterate until termination condition is met.





Dynamic Programming



Bellman's equation: $V(n, S_n) = \min_{\{u_n\}}[r_n + V(n+1, S_{n+1})]$

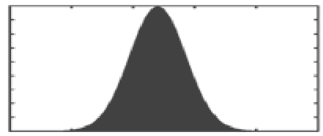
Minimizes the total cost: $\min_{\{u_n\}} \sum_n r_n$

Subject to the transitions: $S_{n+1} = g(n, S_n, u_n)$

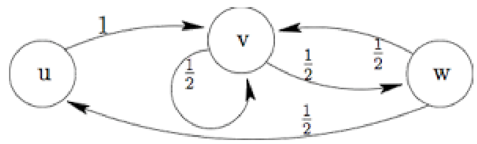
Part 3: Uncertainty

May 16 **Probability**.
- Independence vs. Dependence
- Combinatorics
- Bayes Rule

May 21 **Random Variables and Distributions**. HW6
- Probability functions & correlations
- Binomial, Poisson, Gaussian
- Central limit theorem



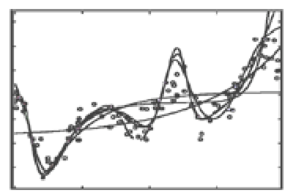
May 23 **Uncertain Dynamics**.
- Stochastic differential equations
- Markov chains



~~May 28 Memorial Day~~, No class

May 30 **Statistics.** HW7
- Hypothesis testing

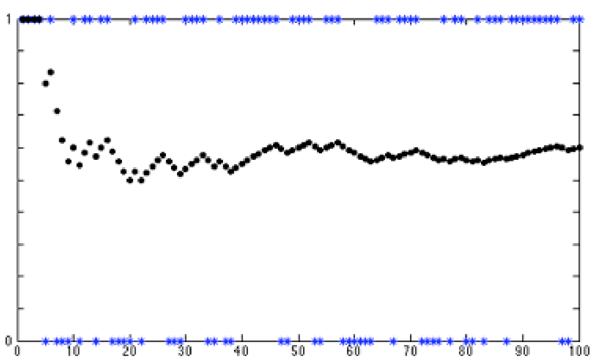
June 4 **Estimation & Bayesian Inference**.
- Maximum likelihood

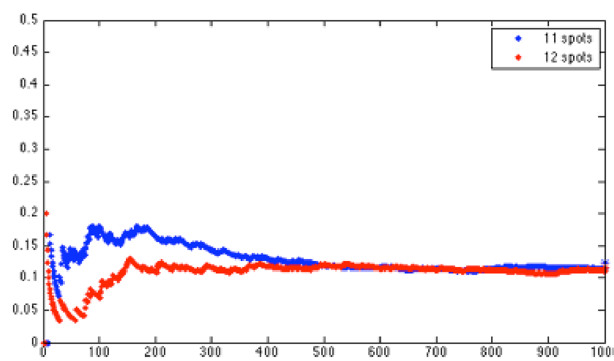


June 6 Review & Practice exam.
June 11 Final Exam: 15:30-17:20

Probability

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$





Conditional Probabilities: $P(A|B) = \dfrac{P(A,B)}{P(A)}$

Random Variables: Map random events to numbers

Probability distribution: $P_\xi(x) = P\{\xi = x\}$

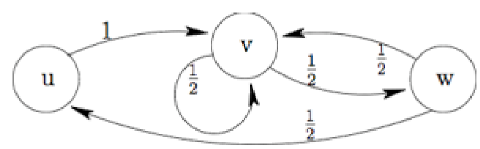
Discrete case: $\sum_{x=-\infty}^{\infty} P_\xi(x) = 1$

Expectation: $\mathbf{E}\xi = \sum_x x P_\xi(x) = \mu$

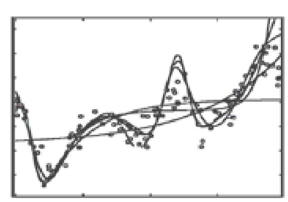
Variance: $\mathbf{D}\xi = \sum_x (x-\mu)^2 P_\xi(x) = \sigma^2$

Applications of Uncertainty

Uncertain Dynamics
- Markov processes
- Stochastic differential equations



Statistics
- Hypothesis testing



Estimation & Bayesian Inference
- Maximum likelihood