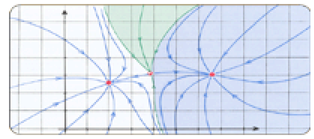
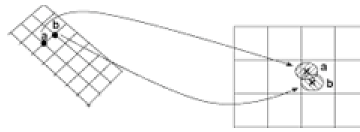
Part 1: Dynamics

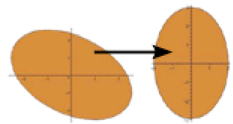
Jan 10 (01) **2-Dimensional flow geometries**. HW1



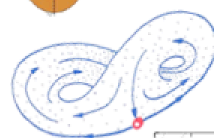
Jan 12 (02) **Discrete dynamics & Mappings**.



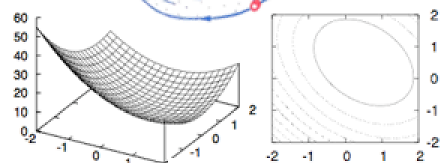
Jan 17 (03) **Diagonalization & eigenvalues**. HW2



Jan 19 (04) **Higher dimensional dynamics & linearization**.



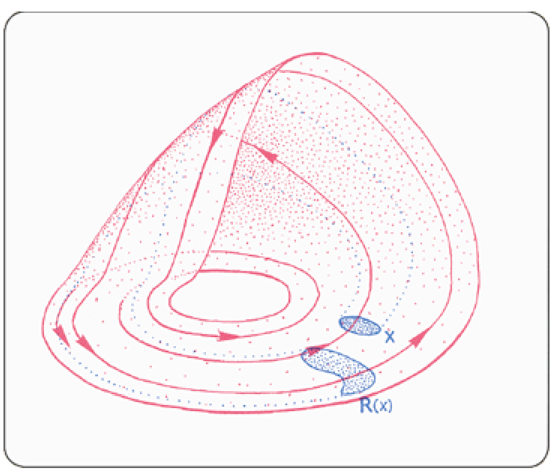
Jan 24 (05) **Stability & Gradient systems**. HW3

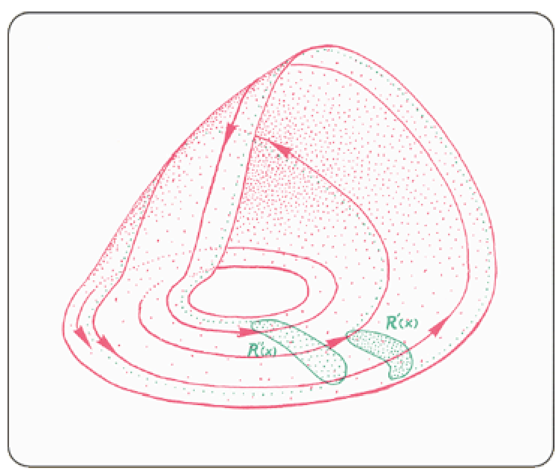


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)
Nonlinear dynamics and chaos, Steven H. Strogatz (1994)
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

01_Dynamics.psd

Expansion of Regions After Each Cycle





Any small error in the measurement of the current state (inevitable) eventually leads to total ignorance of the position of the trajectory within the chaotic attractor.

Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

02_RosslerExpand.psd

Lyapunov Exponent

The Lyapunov exponent is a measure of how much two neighboring initial points will diverge in the dynamics flow.

1-dimensional system: an initial separation, Δx_0.
The separation at a much later time will be given by

$$\Delta x_t = \Delta x_0 e^{\lambda t}$$

Where the Lyapunov exponent of the system is defined by

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln(\Delta x(t))$$

Typo?

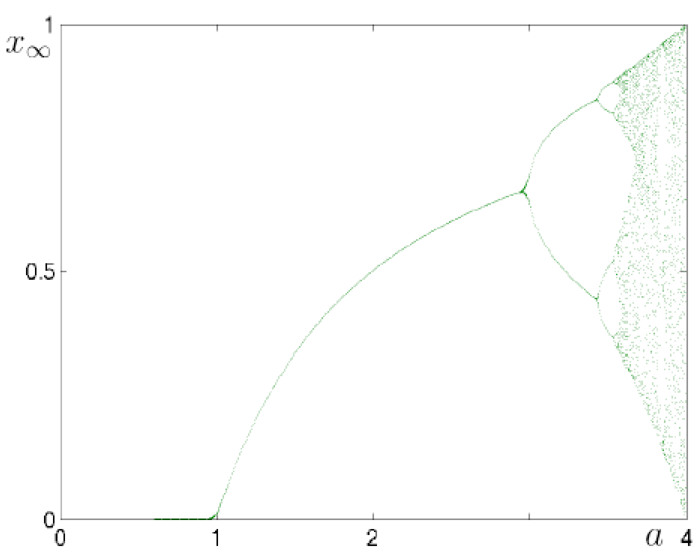
and $\Delta x(t)$ is the average deviation from the unperturbed trajectory.

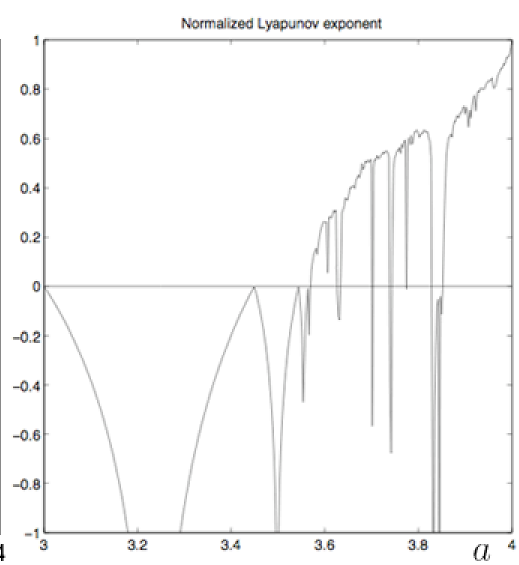
(\frac{\Dx_t}{\Dx_0} ?)

03a_LyExp.psd

Lyapunov Exponent for the Logistic Map

$$\boxed{x_{n+1} = a x_n (1 - x_n)}$$





<http://courses.physics.kth.se/5A1352/>

03b_LyExpLogistic.psd

Lyapunov's Stability Theorem

To show that a system is stable, construct a **Lyapunov function**.

Lyapunov's stability theorem: If there is a Lyapunov function V such that:

$\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f(\bar{x}) = 0$, $\bar{x}$ is a fixed point.

$V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function defined on some neighborhood U of $\bar{x}$.

$V(\bar{x}) = 0$ and $V(x) > 0 \; \forall x \in (U - \bar{x})$.

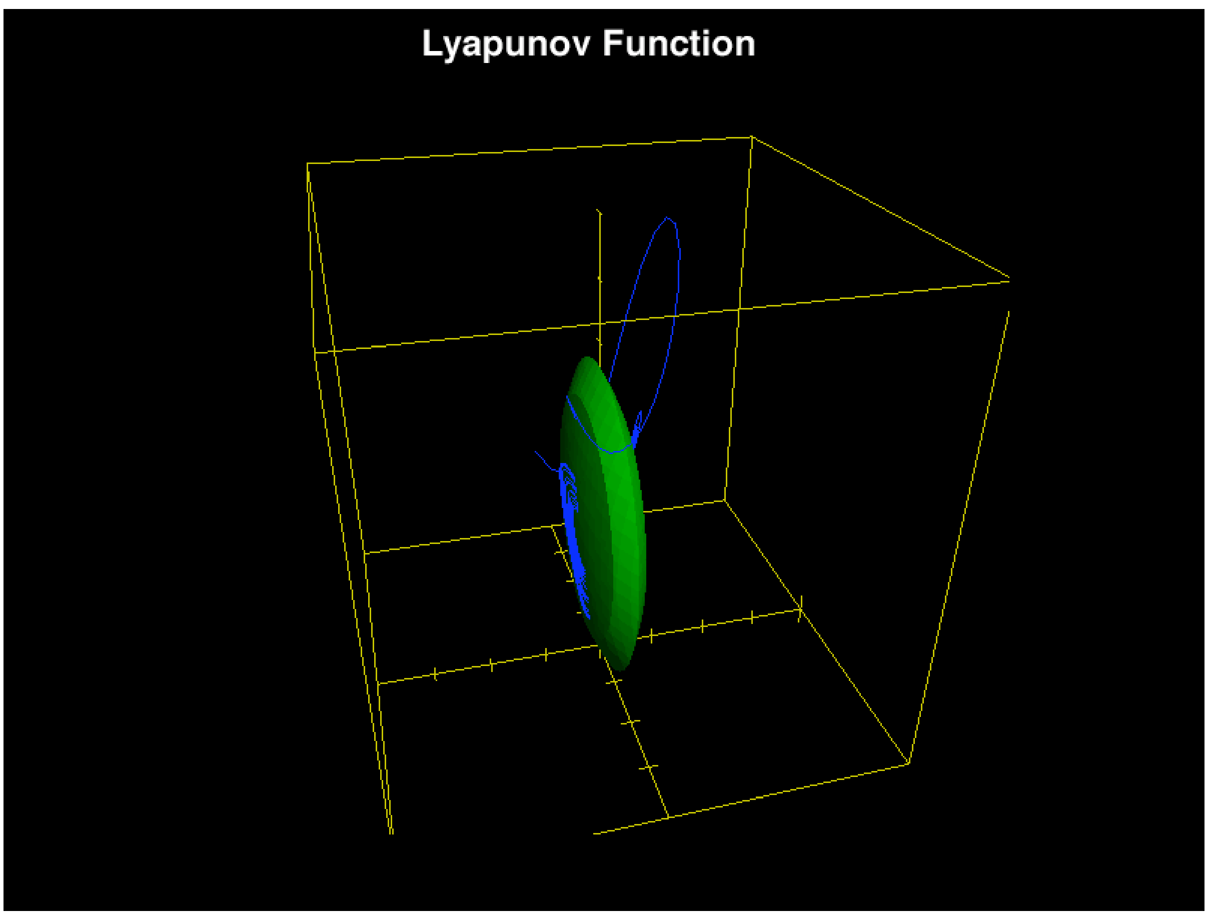
$\dot{V} \leq 0 \; \forall x \in (U - \bar{x})$

then $\bar{x}$ is stable.

If $\dot{V} < 0 \; \forall x \in (U - \bar{x})$, then $\bar{x}$ is asymptotically stable.

(from Andy Fraser's notes)

04_LyapunovFun.psd



05_LyapunovLorenz.psd

Lyapunov Function for Gradient Systems

$$\dot{x} = -\frac{\partial V}{dx} \text{ and } \dot{y} = -\frac{\partial V}{dy}$$

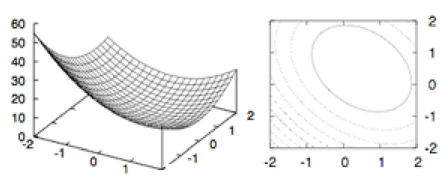
07_LyapunovGradient.psd

Why study gradient systems?

1. Especially “easy” systems to study.
   A generalization of one-dimensional flows.

2. Historically important (physics).
   Many laws of physics can be expressed as gradient systems.

3. Advantagous for applications in optimization problems.
   Convergence theorems for optimization proceedures.



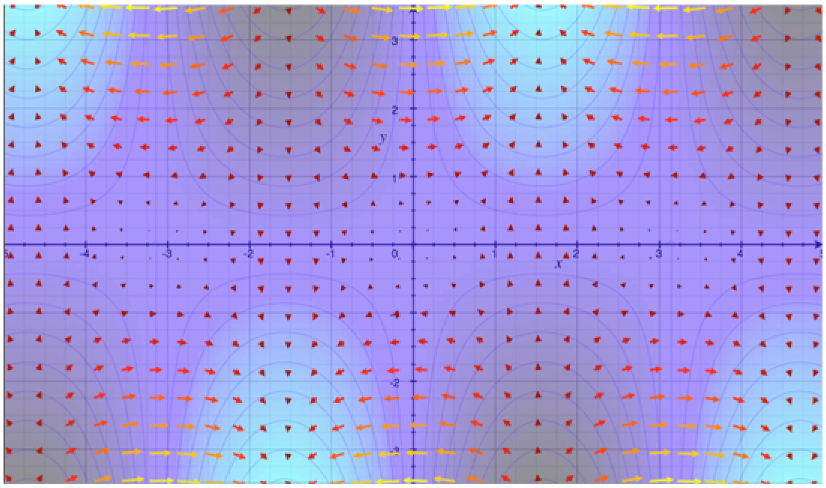
12_WhyGradientSys.psd

Gradient Systems: Gradient of a Potential Functions

Vector fields associated to a scalar potential: $V : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\frac{d}{dt}\vec{x} = \vec{f}(x) = -\nabla V(x)$$

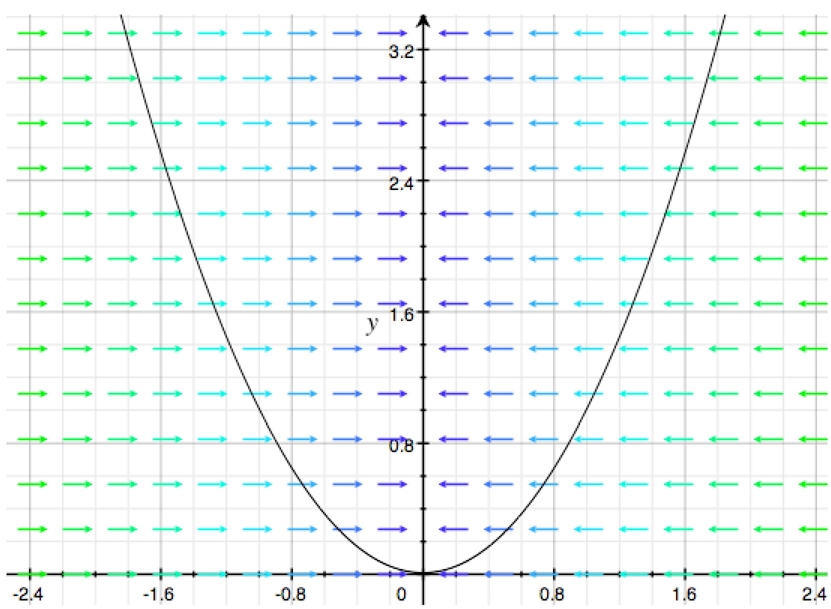
The gradient is the maximal directional derivative: $\nabla V(x) = \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_N \end{bmatrix} V(x)$



$f(x,y) = y \cdot \sin x$

13_WhatGradientSys.psd

Geometry of Gradient Systems: 1-Dimension



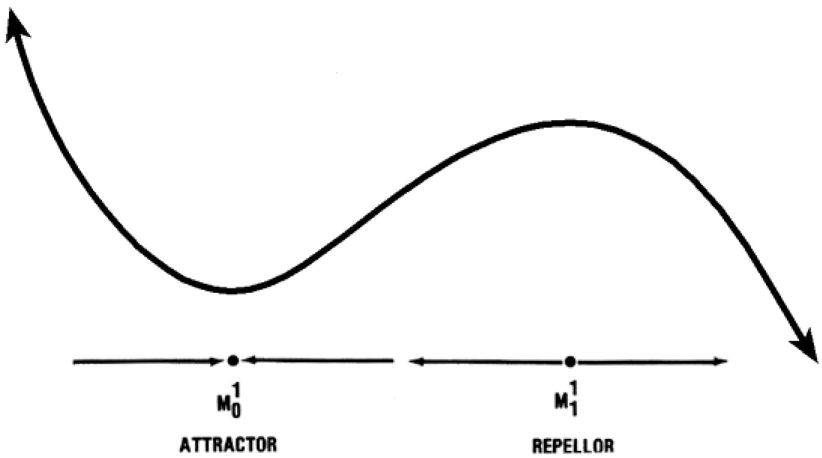
14_1D_levelSets.psd

Attractor versus Repeller: Second Derivative at Fixed Point

The *Hessian* of the potential at each fixed point

$$H_{i,j}(x) \equiv \frac{\partial^2}{\partial x_i \partial x_j} V(x)\Big|_{x=x_0}$$

determines whether the fixed point is an attractor or repeller:

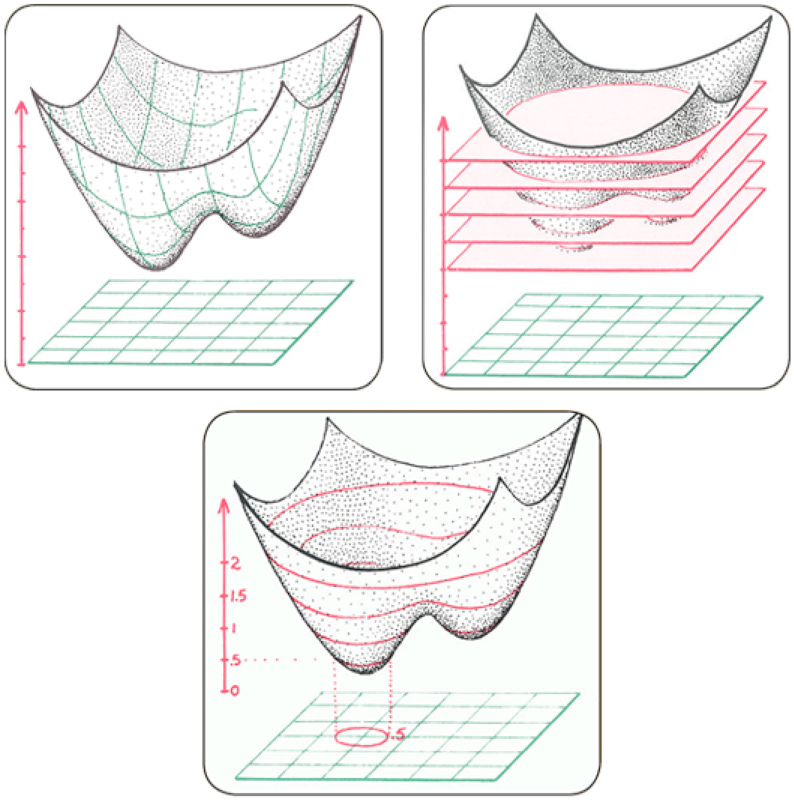


15_criticalValue.psd

Test for Gradient System

16_gradient_test.psd

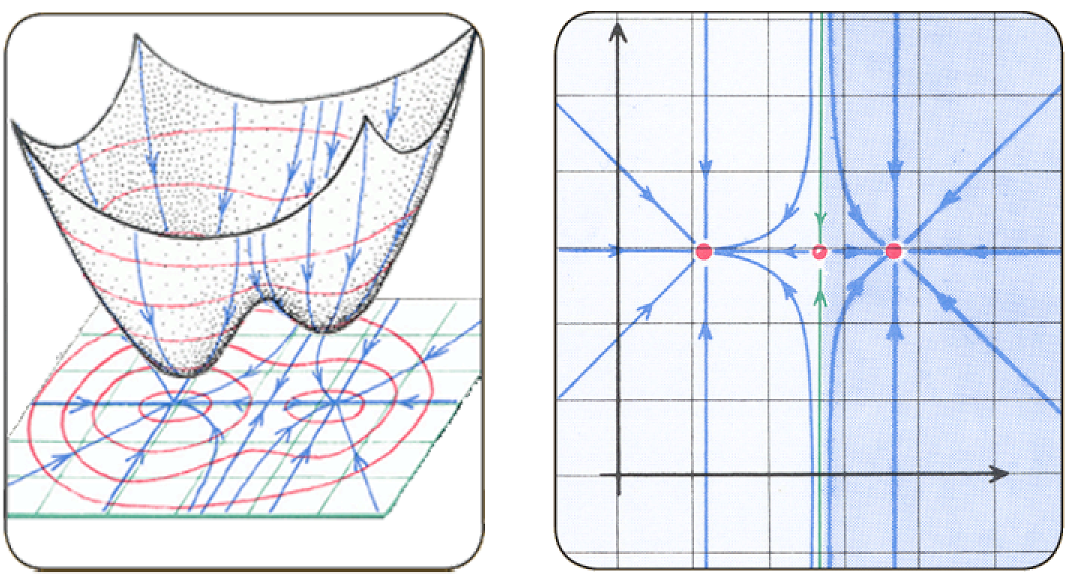
Gradient Systems: Level Sets



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

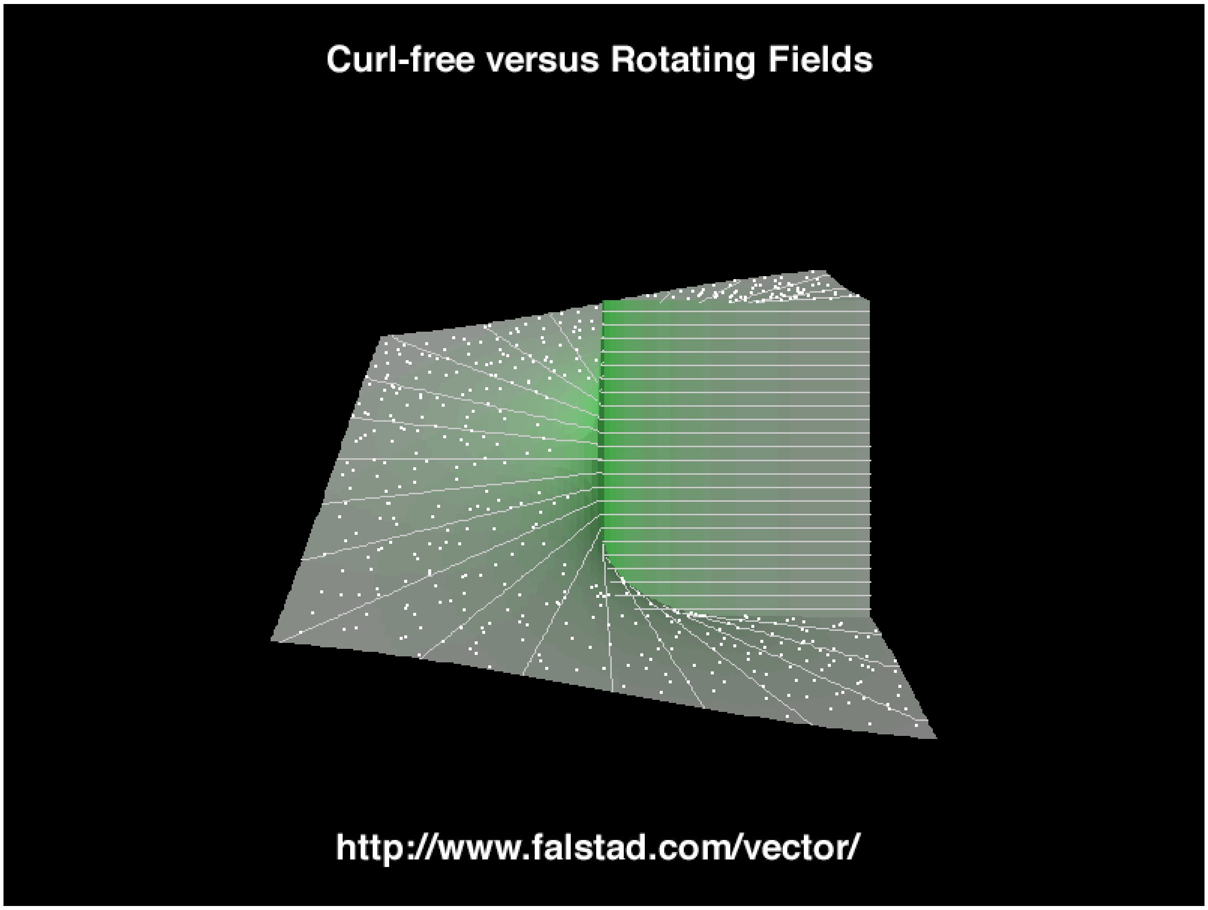
17_levelSets.psd

Gradient Systems: Forces from Potential Function



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

18_potential.psd



19_gradient_test2.psd

Gradient Systems: No Closed Orbits

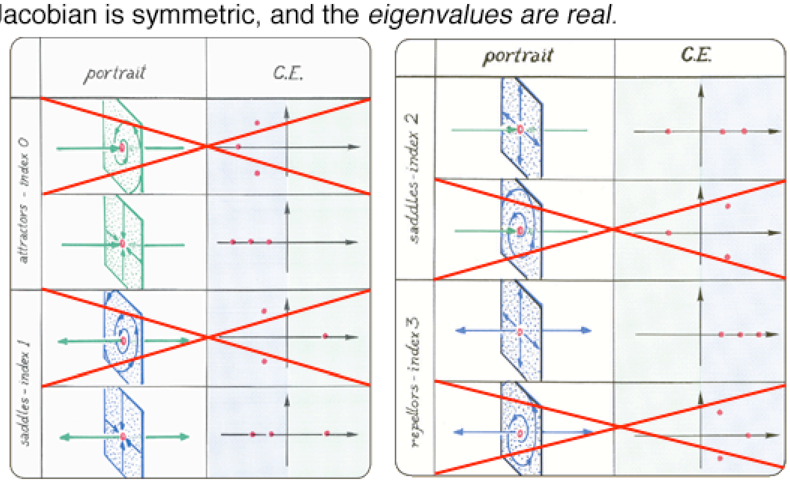
The *Jacobian* of the dynamical system,

$$\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}) = -\nabla V(\vec{x})$$

is the *Hessian* of the potential:

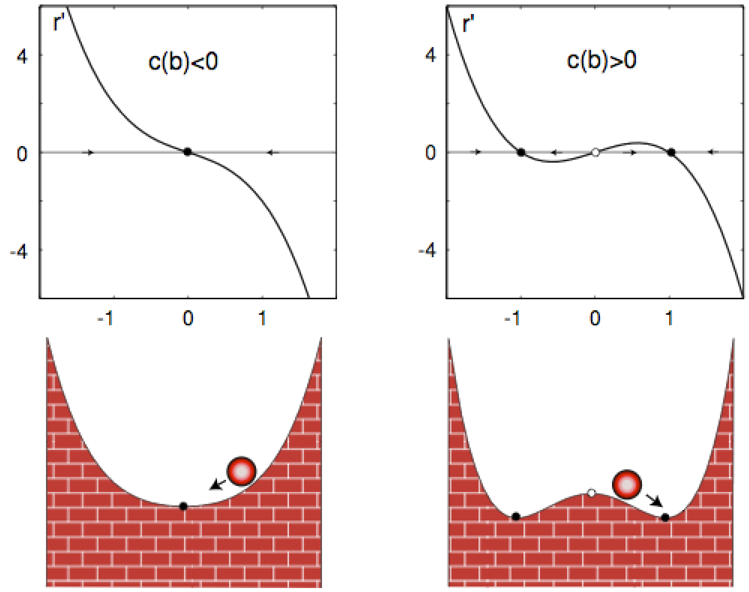
$$H_{i,j}(x) \equiv \frac{\partial^2}{\partial x_i \partial x_j} V(x)$$

Then the Jacobian is symmetric, and the *eigenvalues are real.*



19_NoClosedOrbits.psd

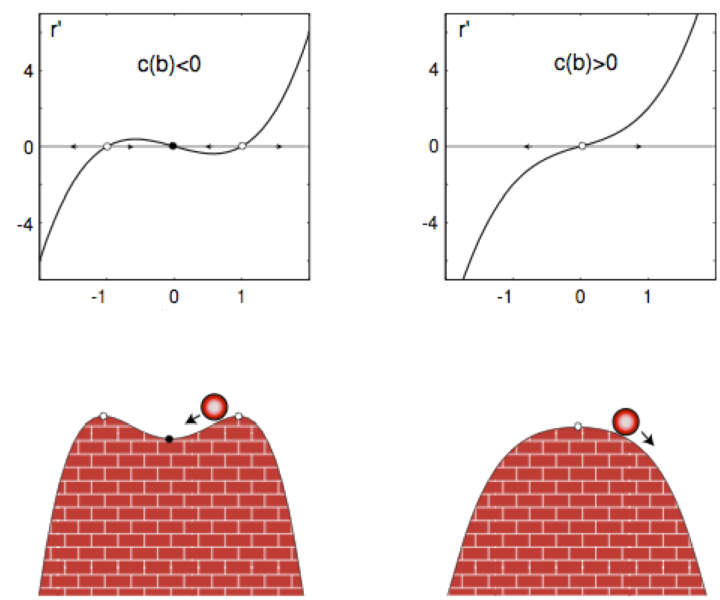
Bifurcations in 1-Dimensional Gradient Systems



Dynamical Systems in Neuroscience (2006) E.M. Izhikevich

20_1dBifur.psd

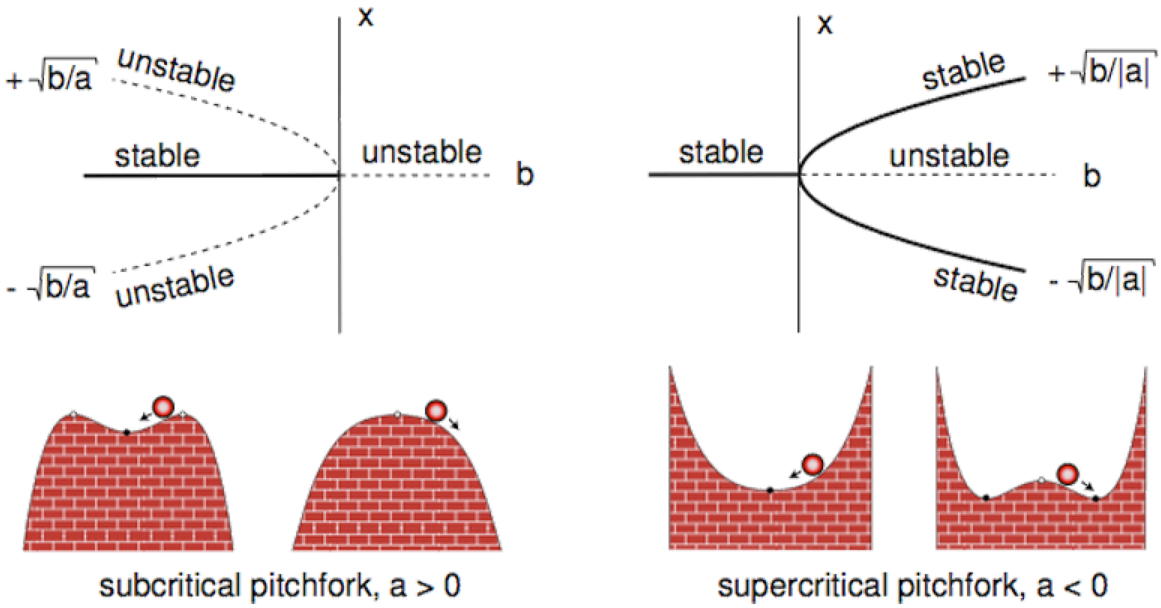
Bifurcations in 1-Dimensional Gradient Systems



Dynamical Systems in Neuroscience (2006) E.M. Izhikevich

21_1dBifur.psd

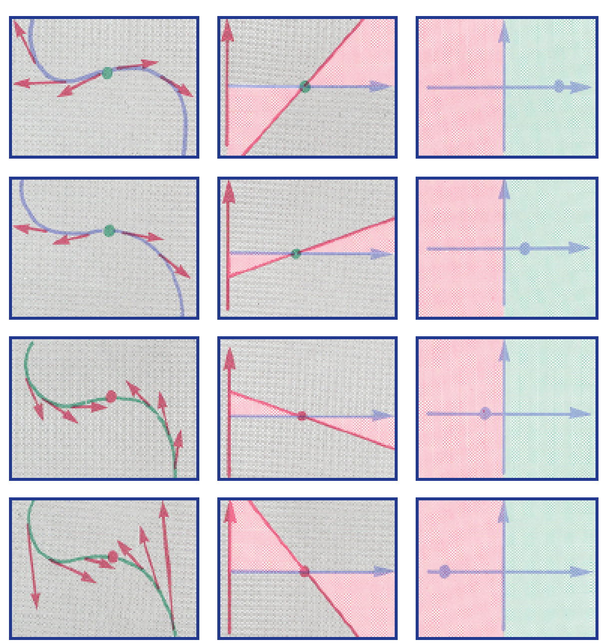
Bifurcations in 1-Dimensional Gradient Systems



Dynamical Systems in Neuroscience (2006) E.M. Izhikevich

22_1dBifur.psd

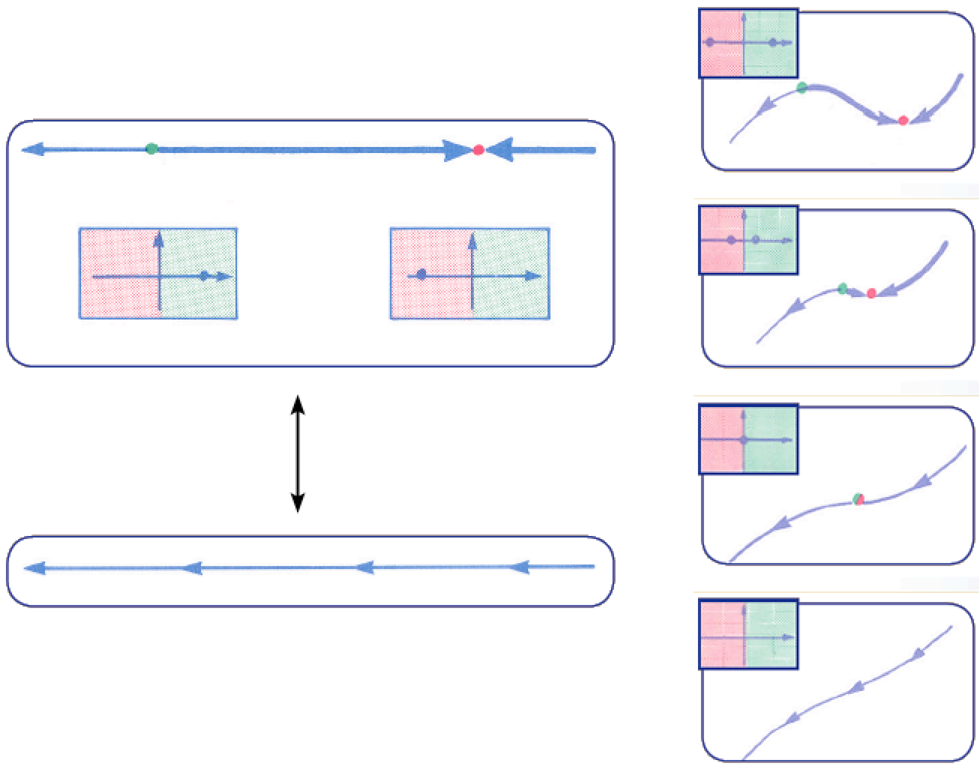
Elementary Catastrophes: Fold



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

23a_fold.psd

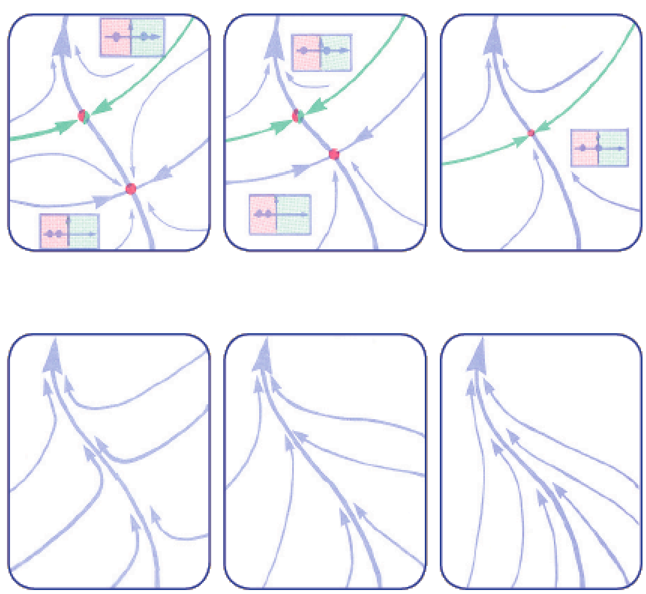
Elementary Catastrophes: Fold



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

23b_fold2.psd

Elementary Catastrophes: 2-D Fold
Saddle-Node Bifurcation



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

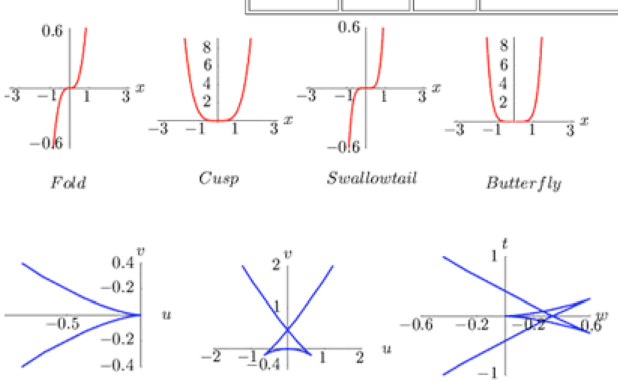
24_2D_fold.psd

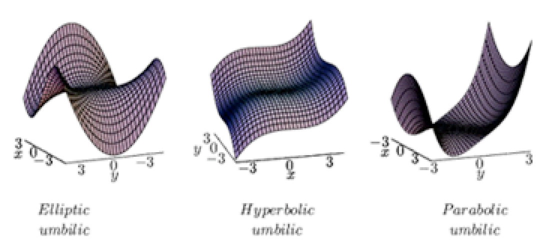
Elementary Catastrophes: 2-D Fold
Saddle-Node Bifurcation

25a_fold.psd

The First Seven Elementary Catastrophes

Germ	Corank	Codim	Universal Unfolding	Name
x^3	1	1	$x^3 + ux$	Fold
x^4	1	2	$x^4 + ux^2 + vx$	Cusp
x^5	1	3	$x^5 + ux^3 + vx^2 + wx$	Swallowtail
x^6	1	4	$x^6 + ux^4 + vx^3 + wx^2 + tx$	Butterfly
$x^3 + y^3$	2	3	$x^3 + y^3 + uxy + vx + wy$	Hyperbolic umbilic
$x^3 - xy^2$	2	3	$x^3 - xy^2 + \underset{-1}{u}(x^2 + y^2) + vx + wy$	Elliptic umbilic
$x^2 + y^4$	2	4	$x^2y + y^4 + ux^2 + vy^2 + wx + ty$	Parabolic umbilic





http://www-wales.ch.cam.ac.uk/~tvb20/elcat.htm

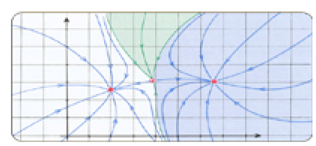
30_unfoldings.psd

Hamiltonian Systems

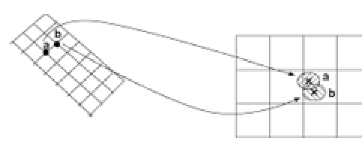
31_hamiltonian.psd

Part 1: Dynamics

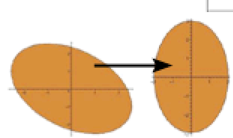
Jan 10 (01) **2-Dimensional flow geometries**. HW1



Jan 12 (02) **Discrete dynamics & Mappings**.

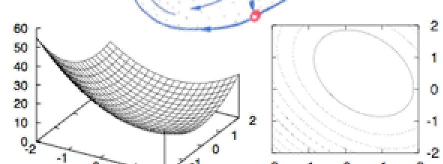


Jan 17 (03) **Diagonalization & eigenvalues**. HW2



Jan 19 (04) **Higher dimensional dynamics & linearization**.

Jan 24 (05) **Stability & Gradient systems**. HW3



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)
Nonlinear dynamics and chaos, Steven H. Strogatz (1994)
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

32_Dynamics.psd