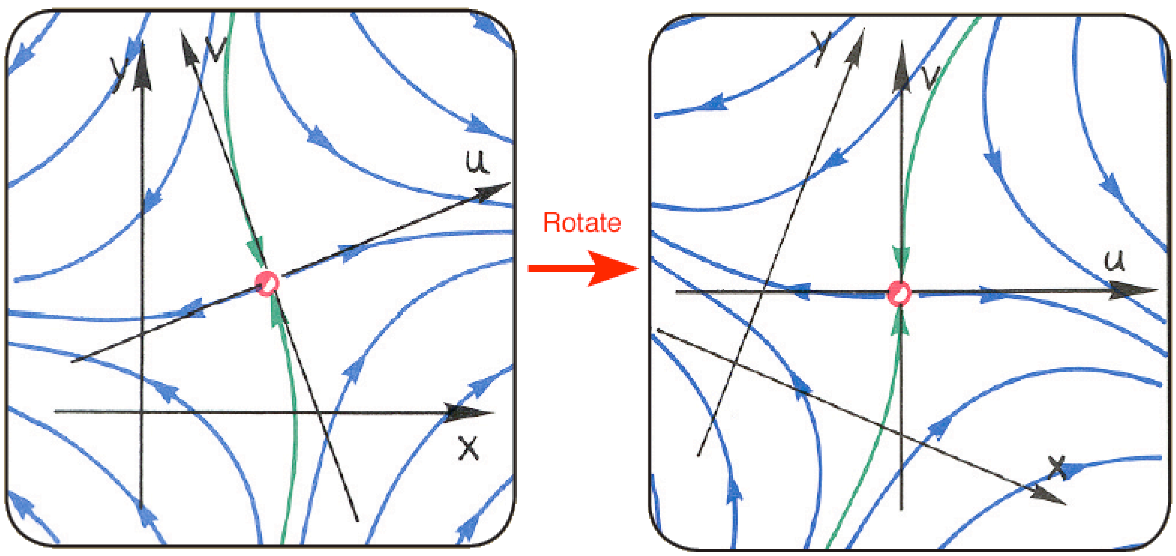
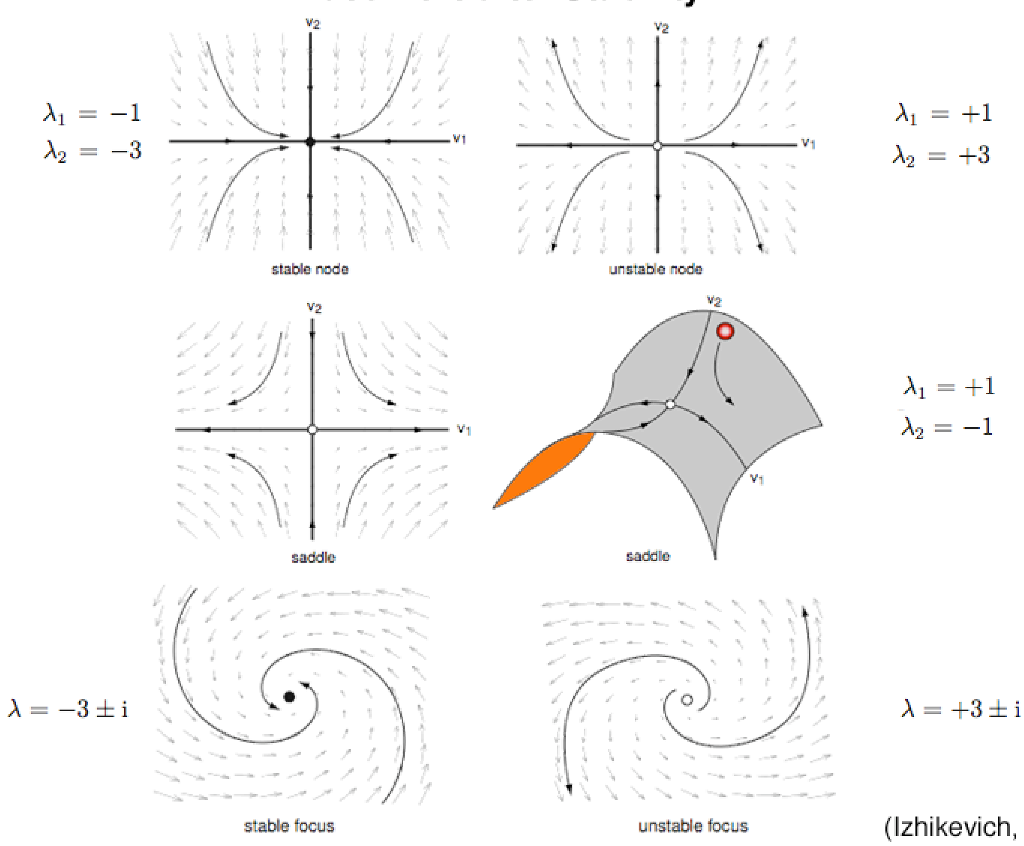
Geometrically: Eigenvectors for Orthonormal Basis



05b_evals.psd

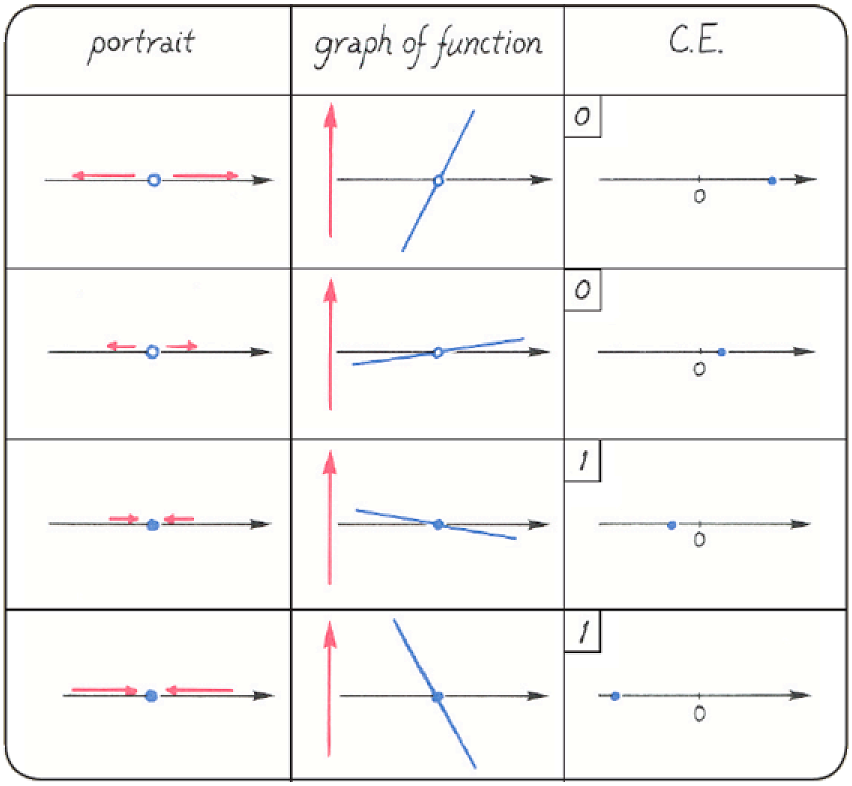
Phase Portraits: Stability



(Izhikevich, 2007)

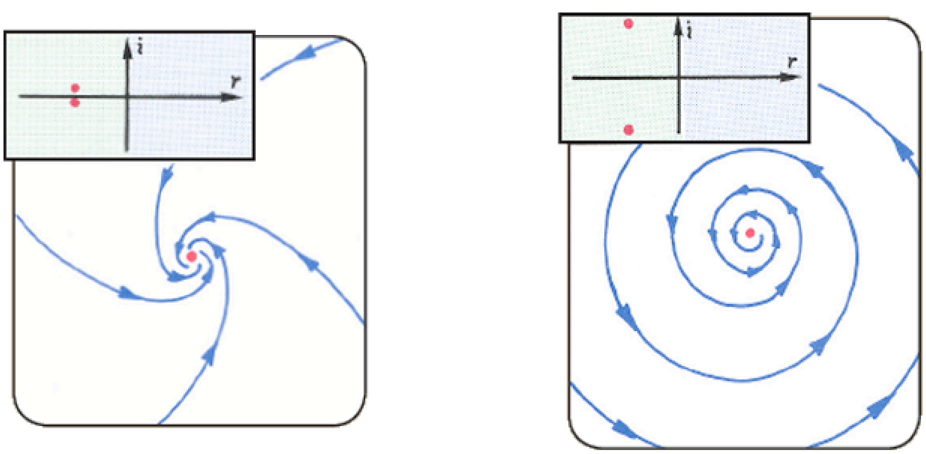
05c_phase stable .psd

One-Dimension
Eigenvalues (Characteristic Exponents) Determine Source/Sink



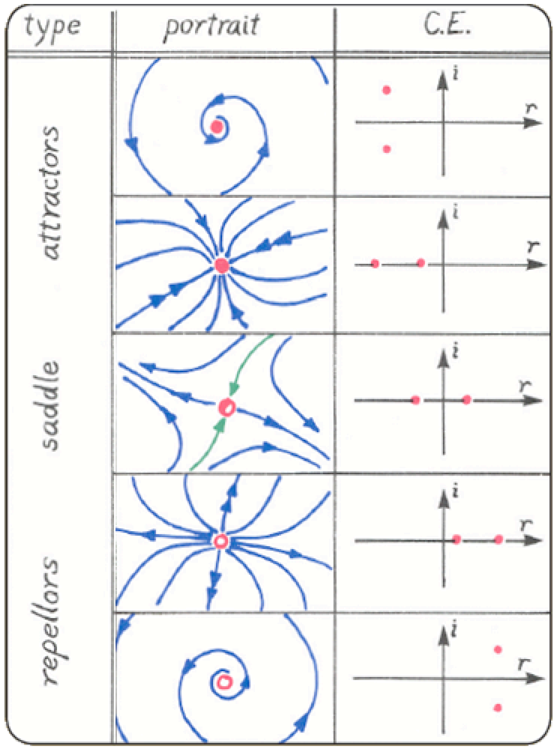
06_1Dim.psd

Two-Dimensions
Imaginary Parts of Eigenvalues Determine Spirals



07_2Dim.psd

Two-Dimensions
Eigenvalues Determine the Geometry of Flows

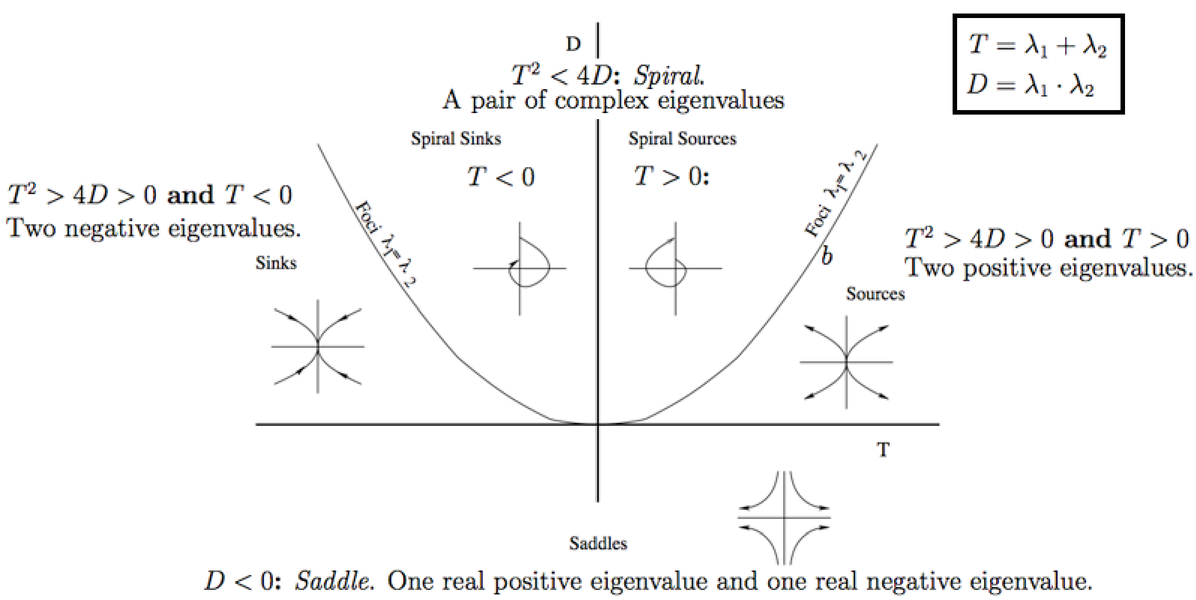


08_peixoto.psd

Great Graph of 2-d Linear Systems

general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with trace $T = a + d$ and determinant $D = ad - cb$:

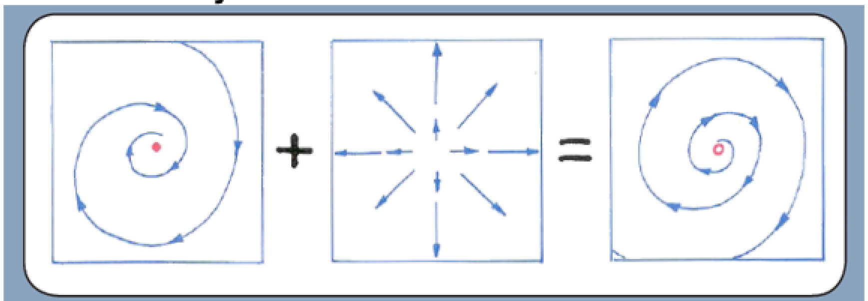
$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - T\lambda + D = 0, \text{ the solutions of which are: } \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$



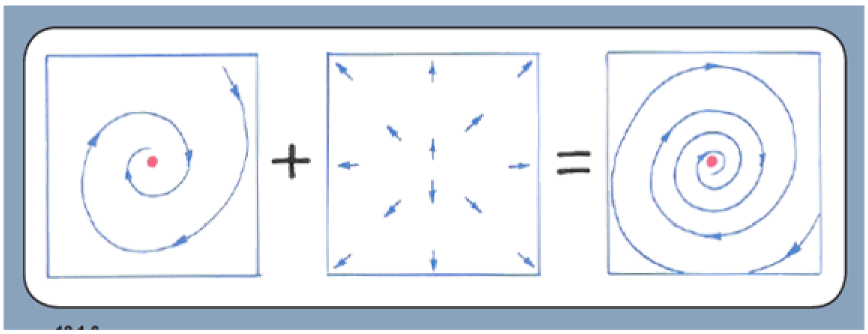
Fraser's notes (2004)

08b_greatGraph.psd

Structural Stability
Flow Geometry is Not Sensitive to Small Perturbations

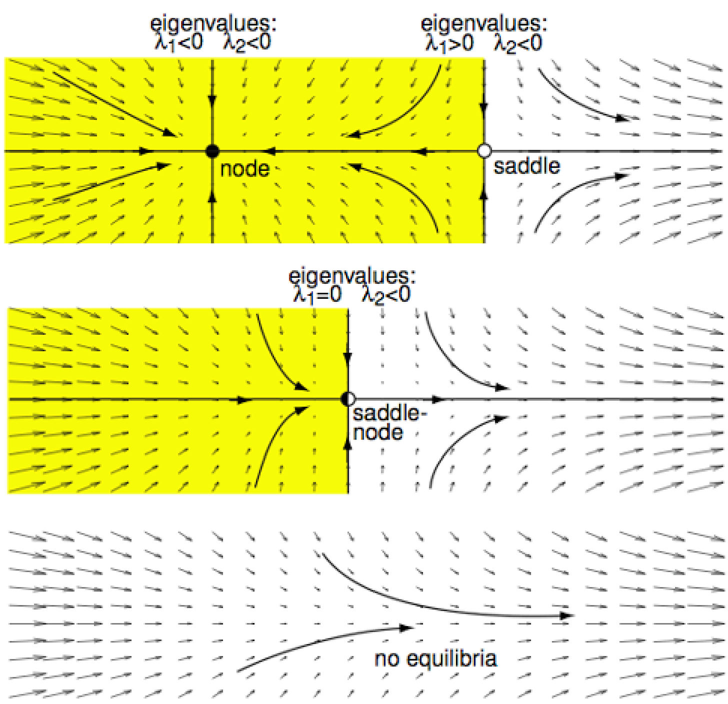


12.1.5.
Imagine a system with a spiral attractor which attracts *very weakly*. By adding a medium-sized perturbation pointing outward, we might be able to change it into a spiral repellor.



09_StructStabil.psd

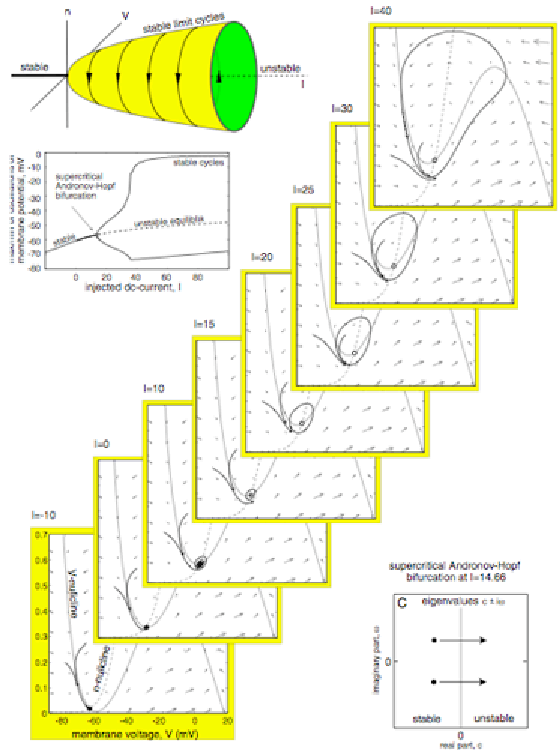
Bifurcations
Parameter Changes that Change Flow Geometry



Saddle-node bifurcation

11b_saddleNode.psd

Bifurcations
Parameter Changes that Change Flow Geometry



Hopf bifurcation

13b_Hopf.psd

Summary: Eigenvalues Determine Flow Geometries

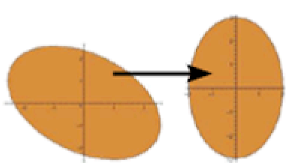
Classify the Dynamics: ***1. Find the fixed points.***

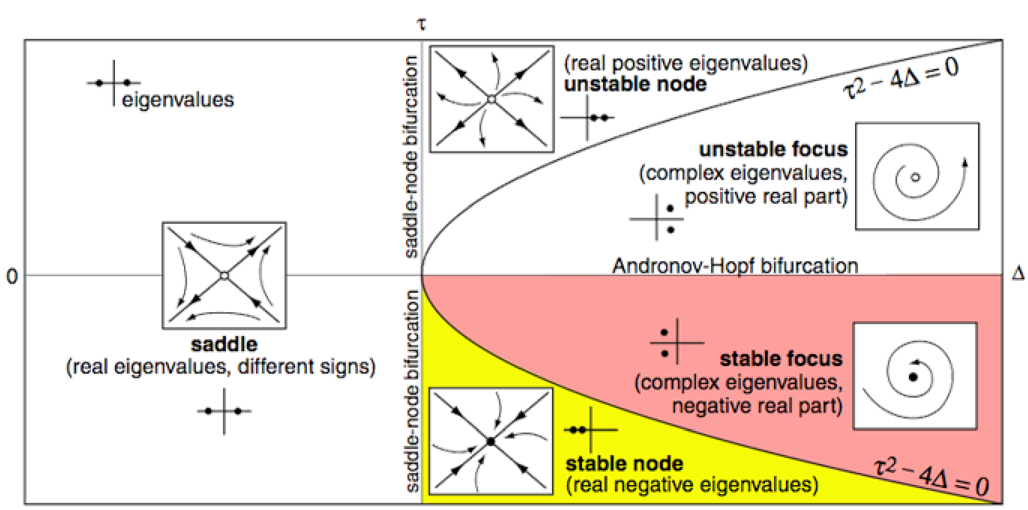
2. Linearize near the fixed points.

3. Compute eigenvalues at fixed points.

4. Classify local stability.

5. Classify bifurcations.

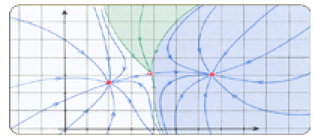




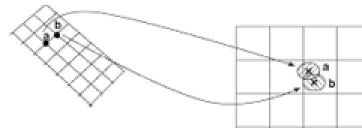
15_summary.psd

Part 1: Dynamics

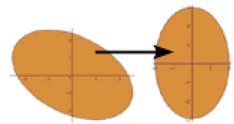
Jan 10 (01) **2-Dimensional flow geometries**. HW1



Jan 12 (02) **Discrete dynamics & Mappings**.

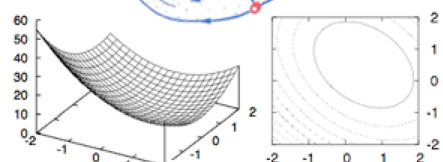


Jan 17 (03) **Diagonalization & eigenvalues**. HW2



Jan 19 (04) **Higher dimensional dynamics**.

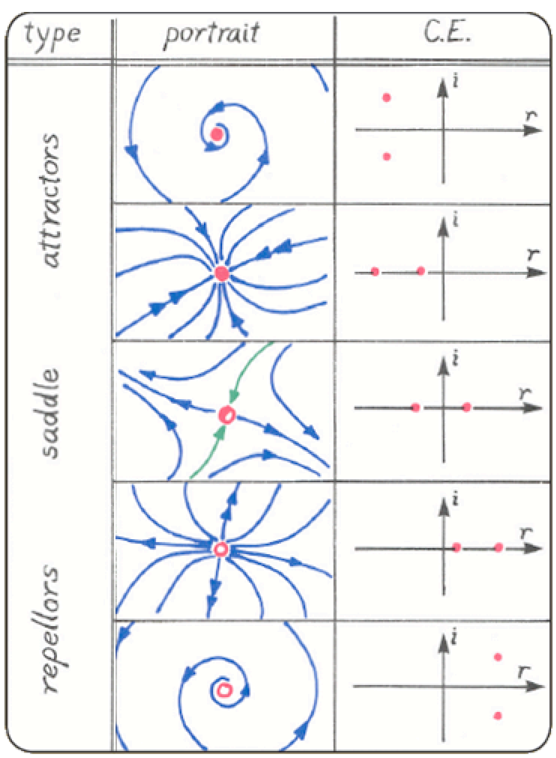
Jan 24 (05) **Stability & Gradient systems**. HW3



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)
Nonlinear dynamics and chaos, Steven H. Strogatz (1994)
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

01_Dynamics.psd

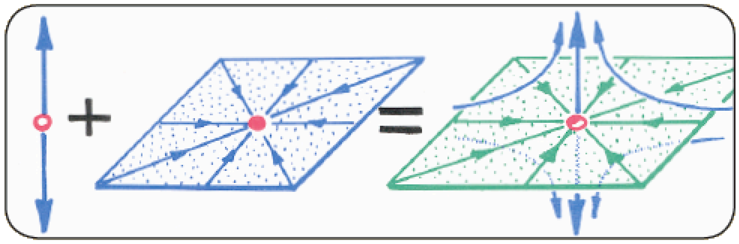
Two-Dimensions
Eigenvalues Determine the Geometry of Flows

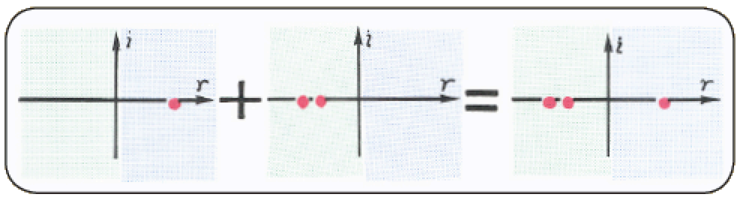


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

02_peixoto.psd

Add a Third Dimension
Complex Geometry of Flows

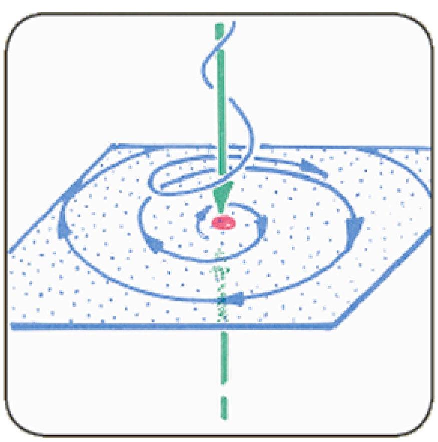


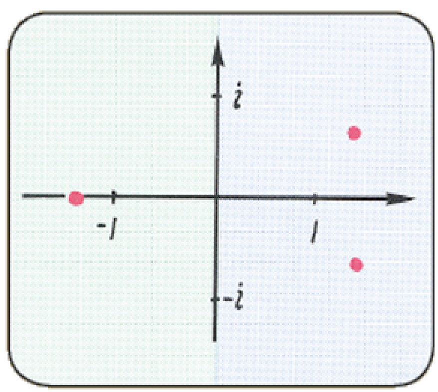


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

03_to3D.psd

Add a Third Dimension
Complex Geometry of Flows: Spiral Saddle

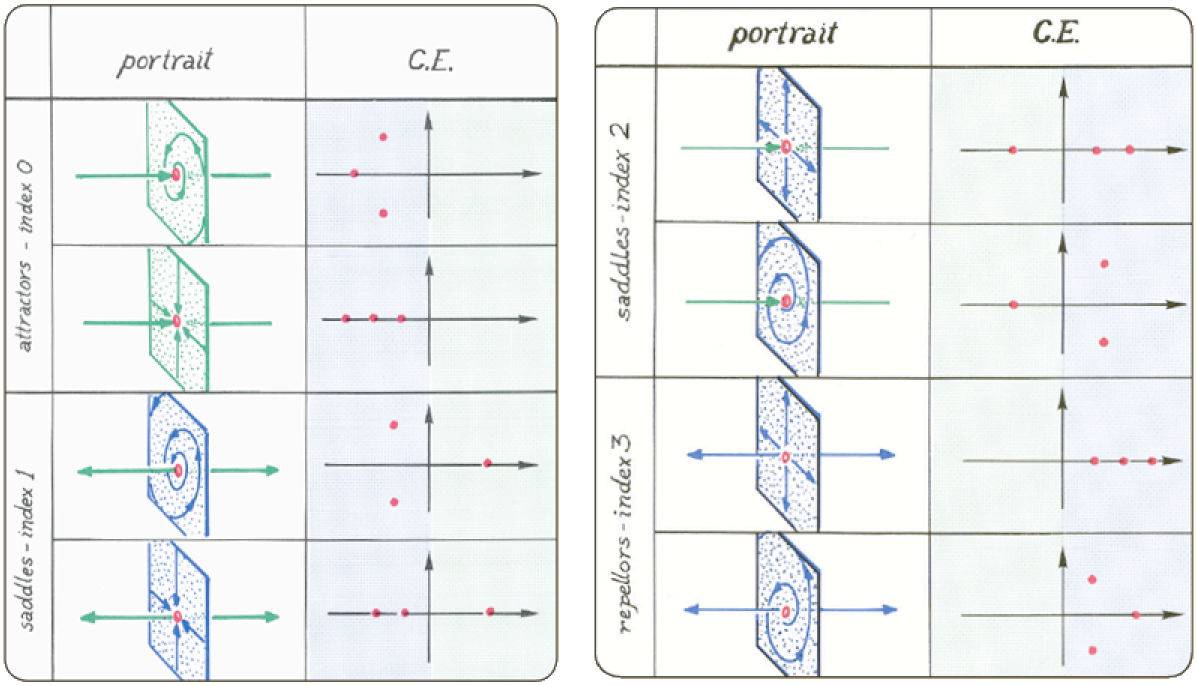




Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

04_to3D2 copy.psd

In Any Dimension
Flow Geometry Follows from Eigenvalues

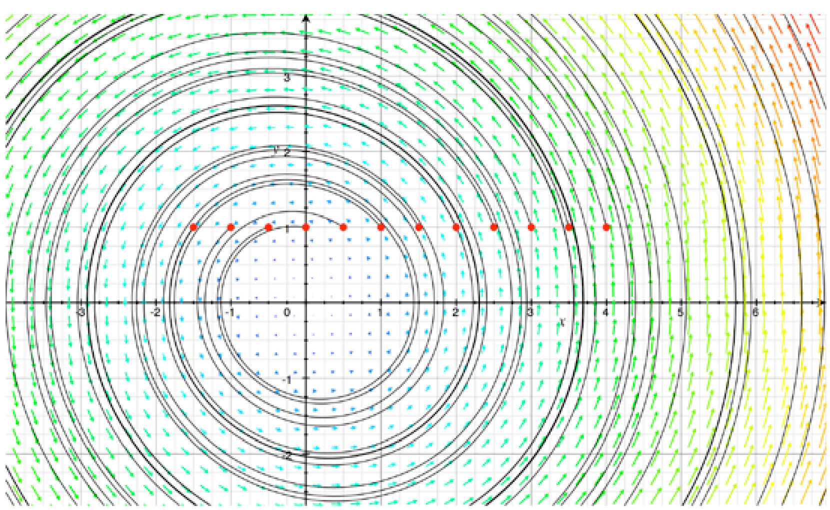


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

05_criticalPts.psd

Slow 2-Dimensional Source

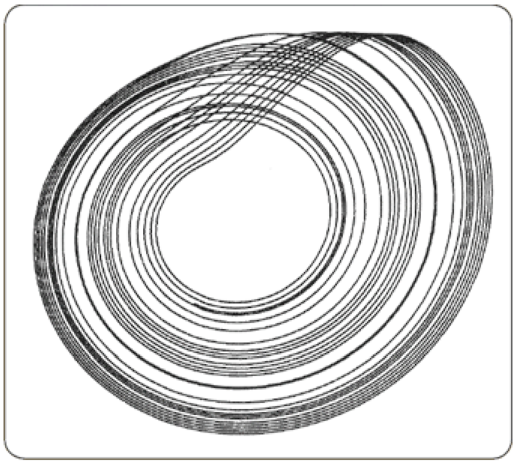
$$\frac{dx(t)}{dt} = -y(t)$$
$$\frac{dy(t)}{dt} = x(t) + 0.15y(t)$$

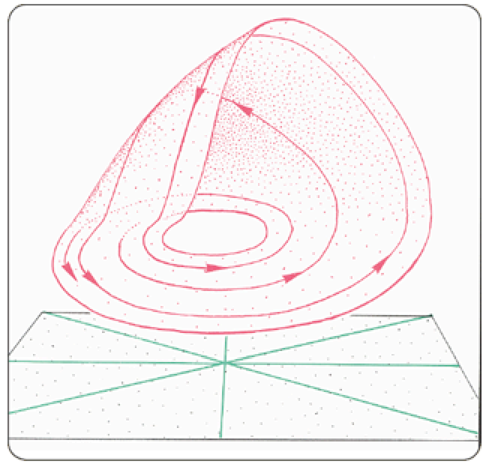


10_slowSource.psd

Add a Third Dimension to Restore

$$\frac{dx(t)}{dt} = -y(t) - z(t)$$
$$\frac{dy(t)}{dt} = x(t) + 0.15y(t)$$
$$\frac{dz(t)}{dt} = b + z(t)(x(t) - c)$$





Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

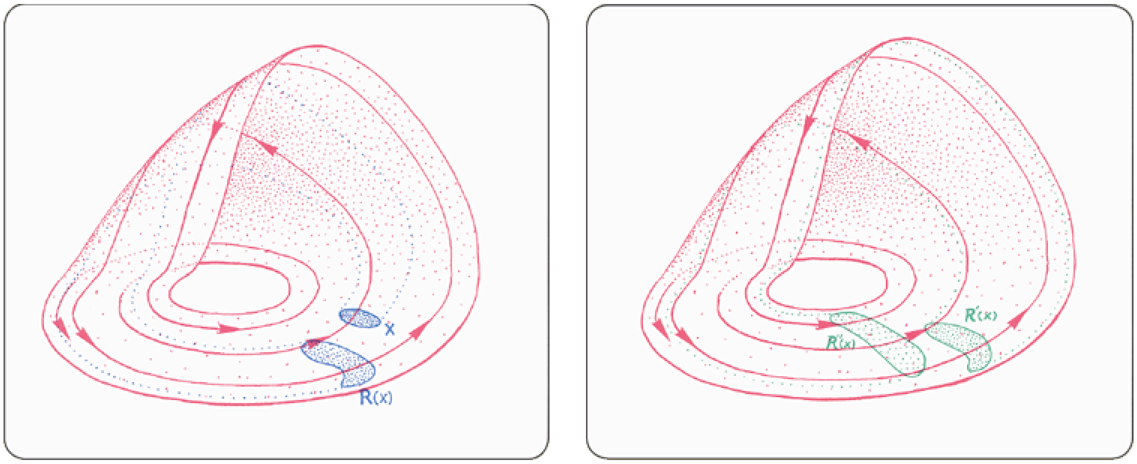
11_Rossler.psd

$$\frac{dx(t)}{dt} = -y(t) - z(t)$$
$$\frac{dy(t)}{dt} = x(t) + 0.15y(t)$$
$$\frac{dz(t)}{dt} = b + z(t)(x(t) - c)$$

RosslerDemo

11_Rossler2.psd

Expansion of Regions After Each Cycle

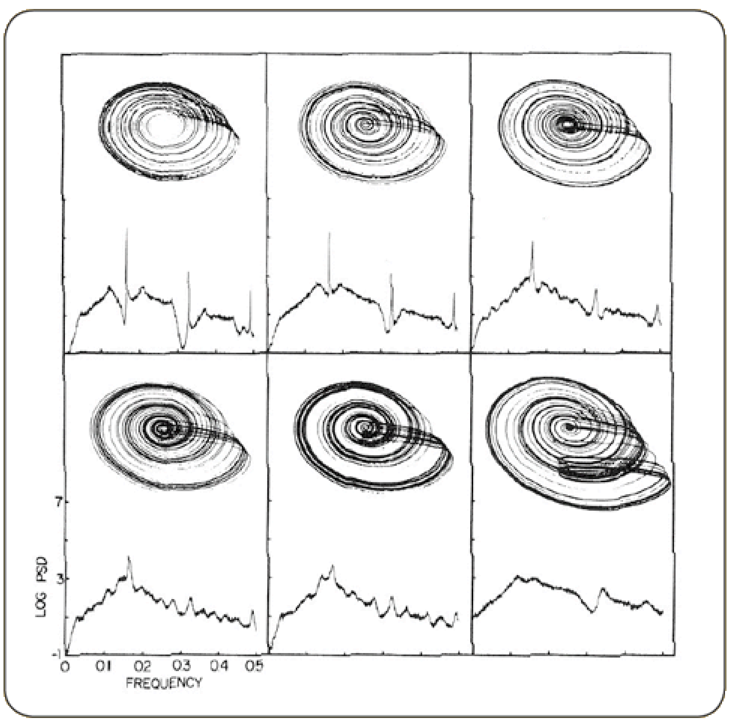


Any small error in the measurement of the current state (inevitable) eventually leads to total ignorance of the position of the trajectory within the chaotic attractor.

Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

12_RosslerExpand.psd

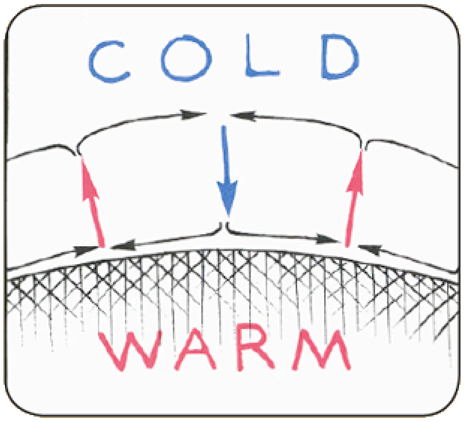
In Dynamical Chaos, Noise can be Signal

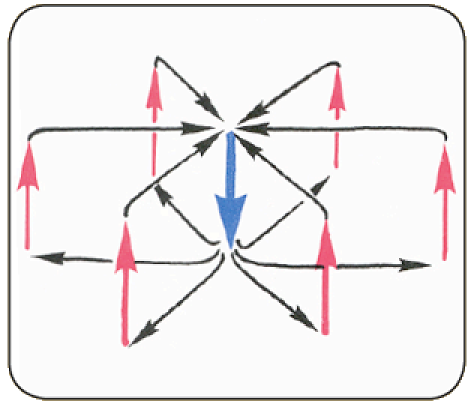


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

13_RosslerFreq.psd

Atmospheric Dynamics & Turbulence





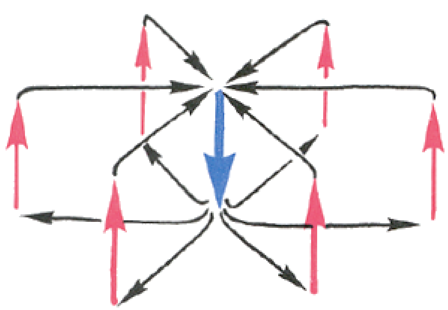
The traffic problem of the competing warm and cold air masses is solved by circulation vortices, called Bénard cells.

In three dimensions, a typical vortex may have warm air rising in a ring and cool air descending in the center.

Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

14_lorenzMotive.psd

Lorenz Equation (1963) Simplifies the Dynamics of a Single Cell



$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(R - z) - y \\ \dot{z} &= xy - bz \end{aligned}$$

$\sigma = 10,\ R = 28,\ b = 8/3$

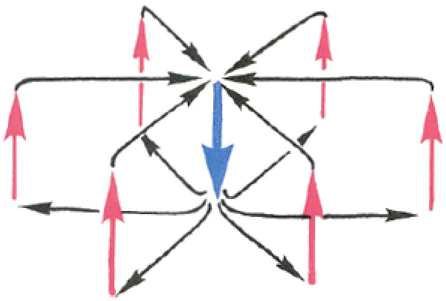
x = *rate of convection overturning*
y = *horozontal temperature gradient*
z = *vertical temperature gradient*

Ed Lorenz's computer simulations showed that the trajectories of solutions have a sensitive dependence on initial conditions.

The three parameter are positive and are the Prandtl number, the Rayleigh number, and a scaling factor.

15_lorenzEq.psd

Lorenz Equation (1963) Simplifies the Dynamics of a Single Cell



$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(R-z)-y\\ \dot{z} &= xy-bz\end{aligned}$$

$\sigma = 10,\ R = 28,\ b = 8/3$

x = rate of convection overturning
y = horozontal temperature gradient
z = vertical temperature gradient

Ed Lorenz's computer simulations showed that the trajectories of solutions have a sensitive dependence on initial conditions.

The three parameter are positive and are the Prandtl number, the Rayleigh number, and a scaling factor.

16_lorenzDyn.psd

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(R-z)-y\\ \dot{z} &= xy-bz\end{aligned}$$

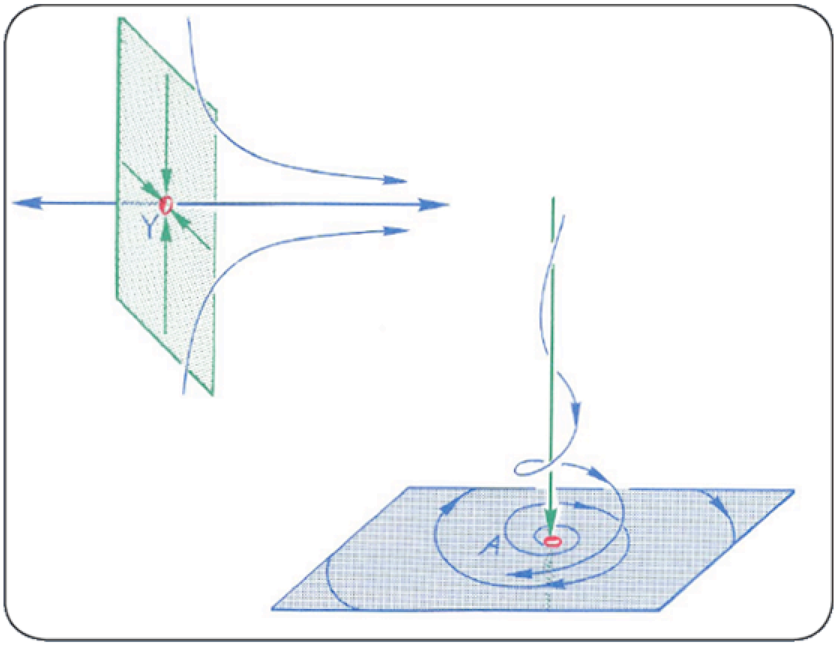
$\sigma = 10,\ R = 28,\ b = 8/3$

lorenzDemo.m

16_lorenzDyn2.psd

Trajectories of the Lorenz System

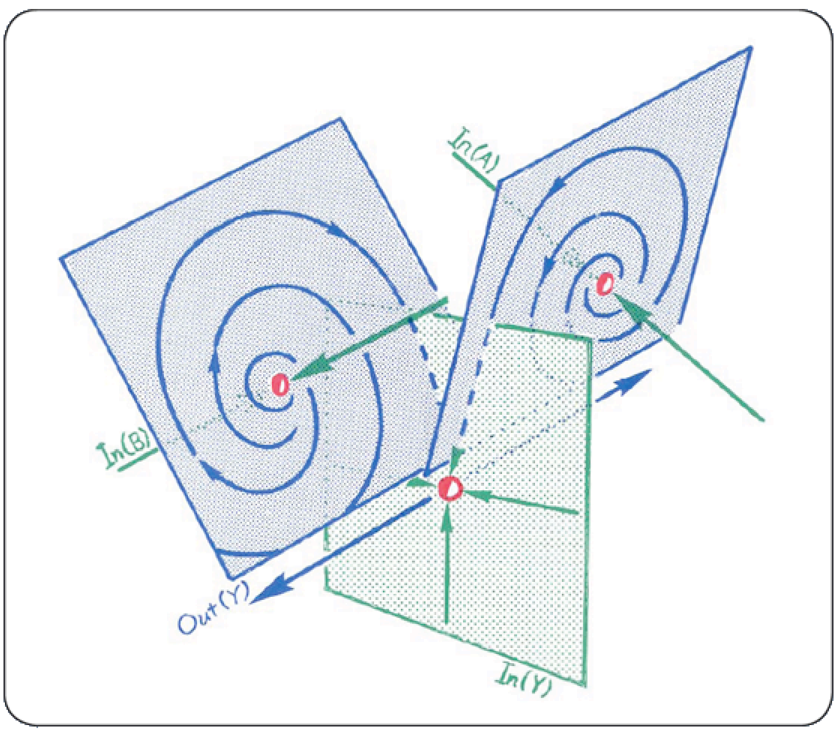
There is a radial saddle point (*receptor*) situated between two spiral saddle points (*donors*).



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

18_lorenzDyn2.psd

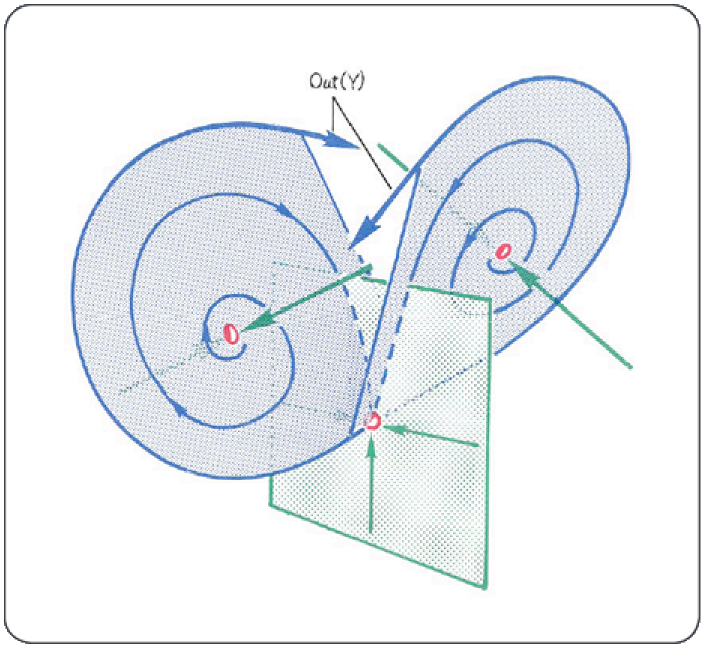
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

19_lorenzDyn3.psd

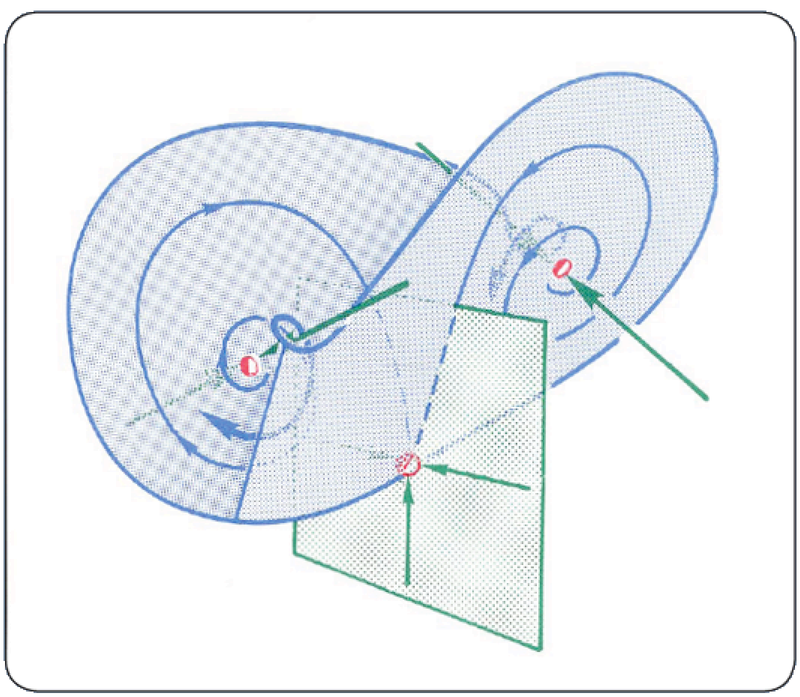
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

20_lorenzDyn4.psd

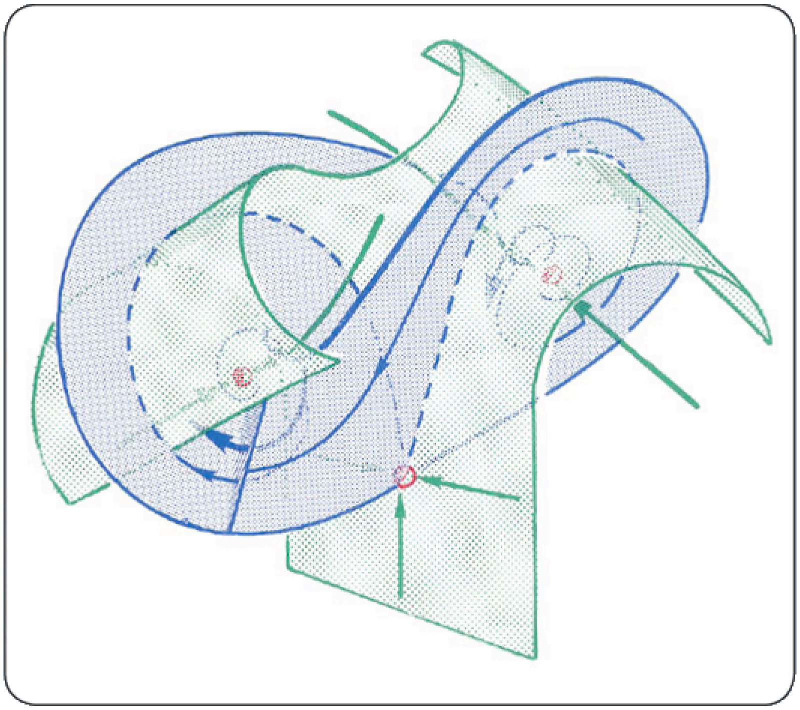
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

21_lorenzDyn5.psd

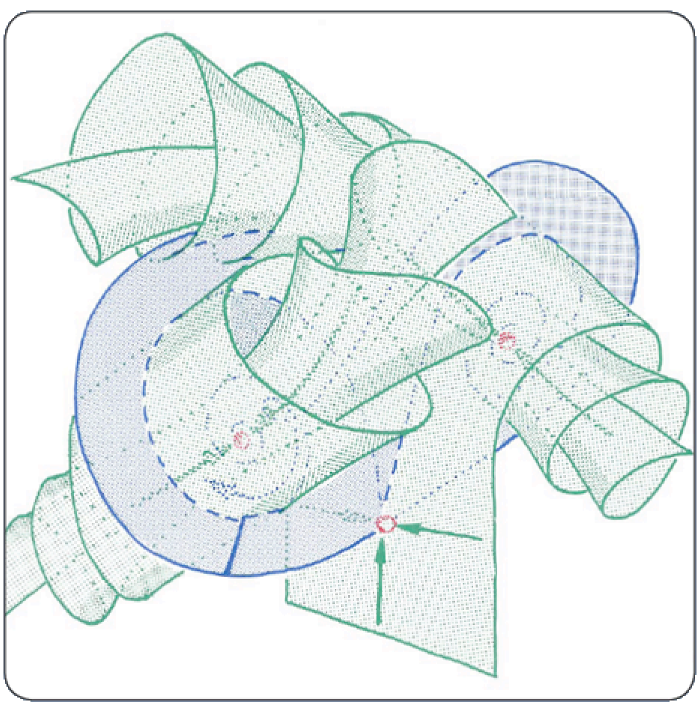
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

22_lorenzDyn6.psd

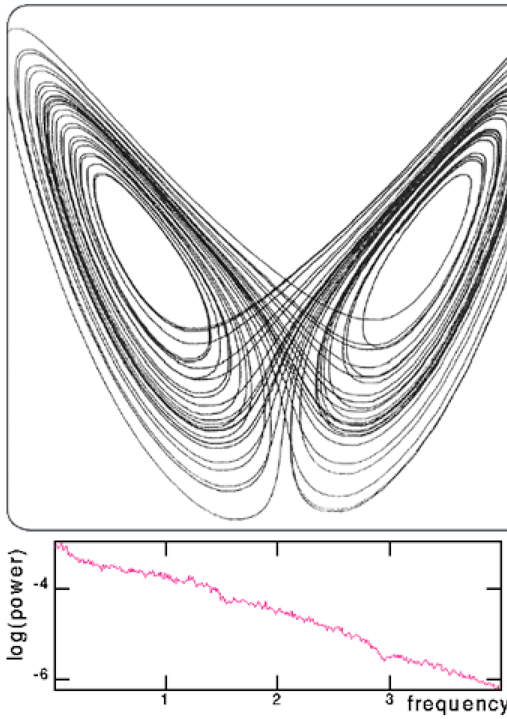
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

23_lorenzDyn7.psd

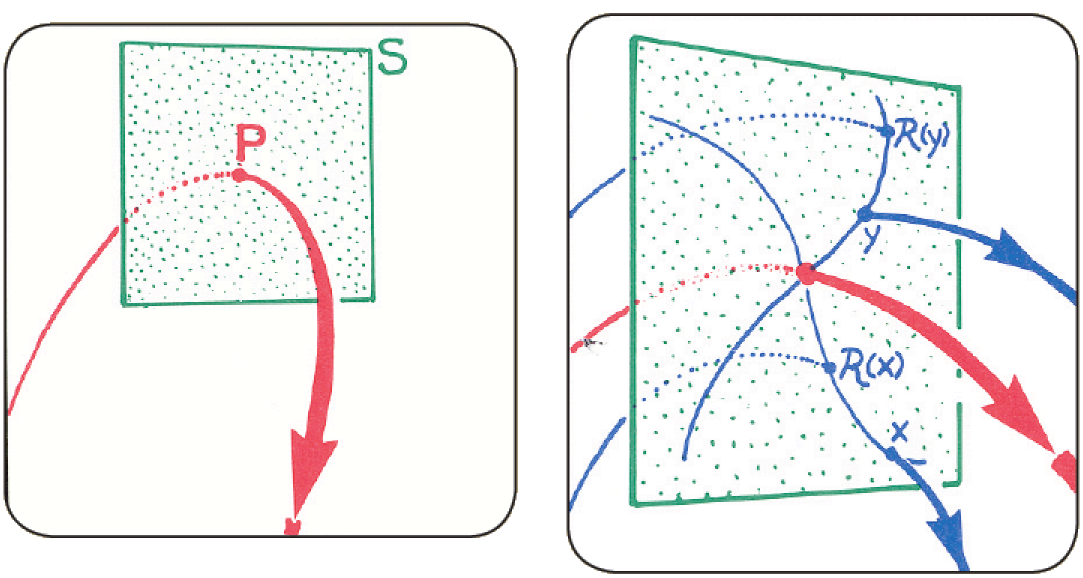
Trajectories of the Lorenz System



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

24_lorenzDyn8.psd

Poincaré Sections: Snap-shot View of Dynamics

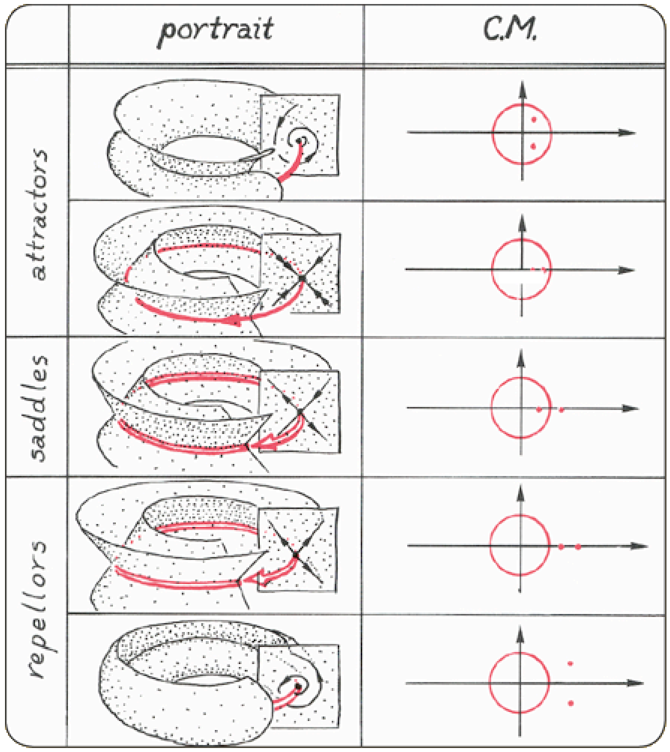


The Poincaré section is two-dimensional, like a strobe-flash at each cycle.

Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

25_poincareSections.psd

Poincaré Section is a Type of Dimensional Reduction

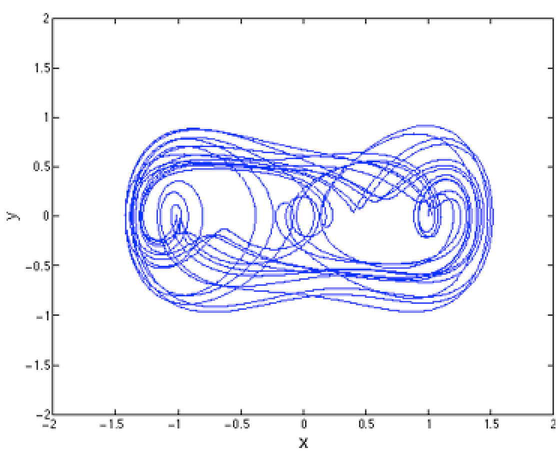


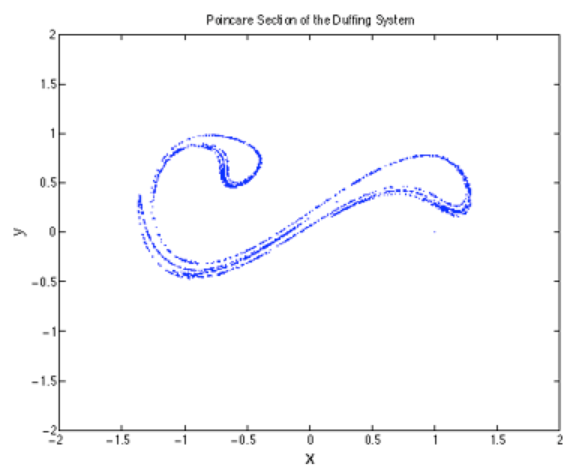
Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

26_poincareDimReduce.psd

The Duffing System:
An example of a non-autonomous (time-dependent, forced) system.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - ky - x^3 + \Gamma\cos(\omega t) \end{aligned}$$



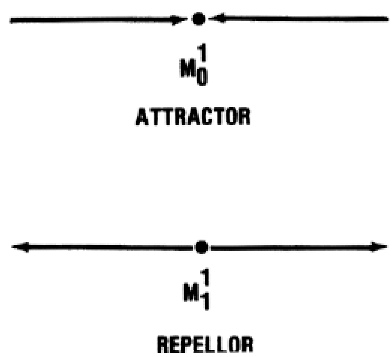


The Poincaré map can be studied as a discrete system to classify the dynamics.

28_duffing.psd

Quantify a System's Dynamics:
Step #4: *Classify the Dynamics*

$$\dot{x} = ax$$

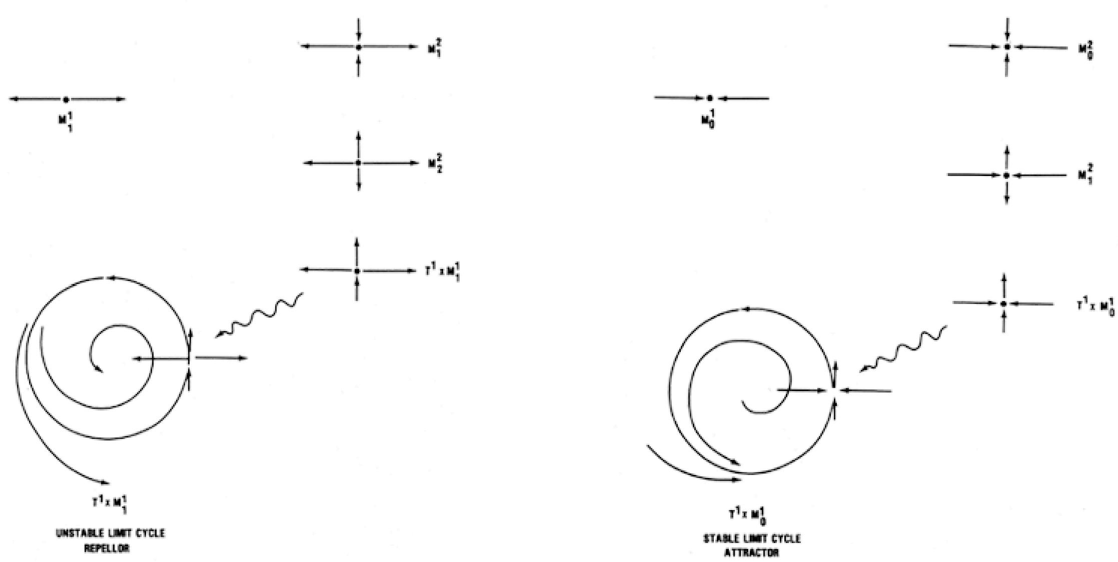


Catastrophe Theory for Scientists and Engineers (1996) R. Gilmore

31_classify1D.psd

Quantify a System's Dynamics:
Step #4: *Classify the Dynamics*

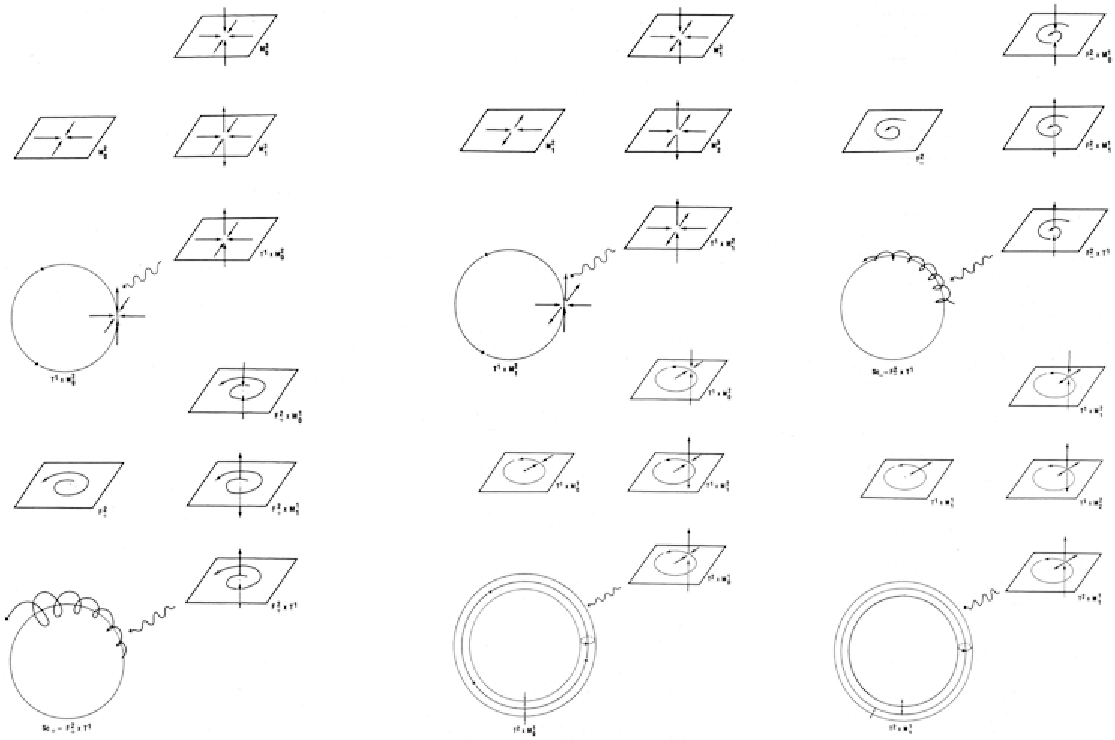
$$\frac{d}{dt}\begin{bmatrix} R(t) \\ J(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} R_0 - R(t) \\ J_0 - J(t) \end{bmatrix}$$



Catastrophe Theory for Scientists and Engineers (1996) R. Gilmore

32_classify2D.psd

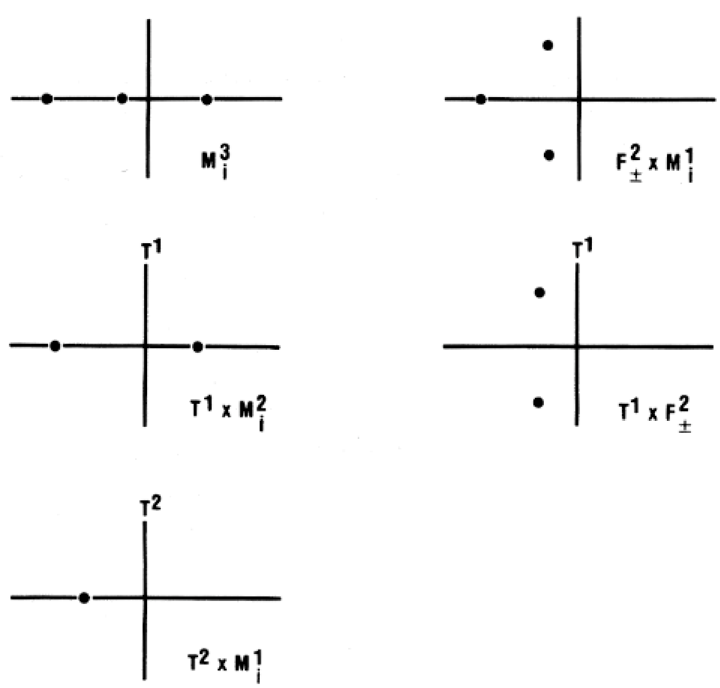
Quantify a System's Dynamics:
Step #4: *Classify the Dynamics: 3-Dimensions*



Catastrophe Theory for Scientists and Engineers (1996) R. Gilmore

33_classify3D.psd

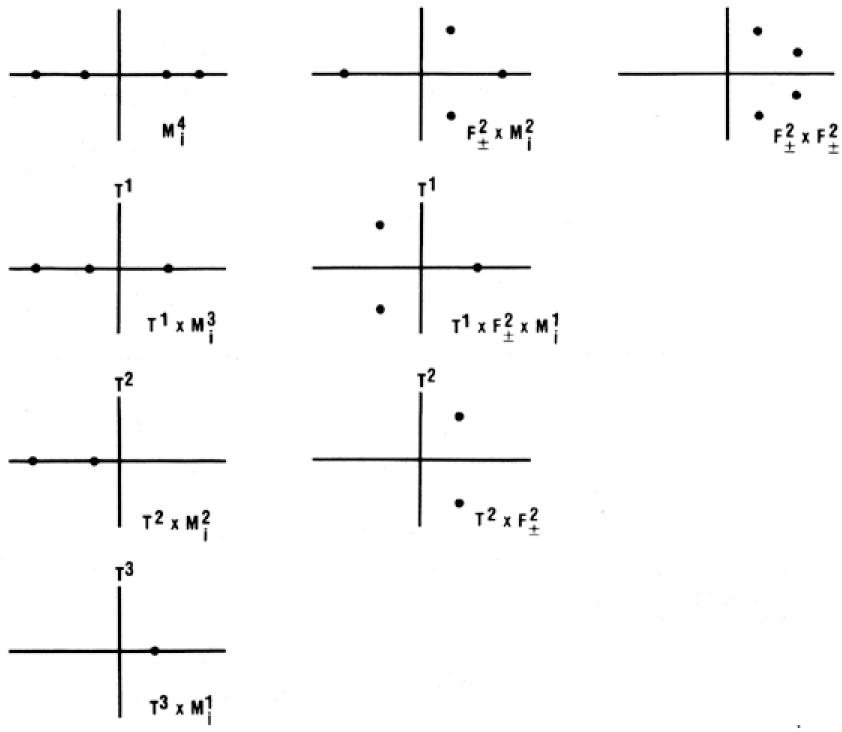
Quantify a System's Dynamics:
Step #4: *Classify the Dynamics: 3-Dimensions*



Catastrophe Theory for Scientists and Engineers (1996) R. Gilmore

34_classify3D.psd

Quantify a System's Dynamics:
Step #4: *Classify the Dynamics: 4-Dimensions*

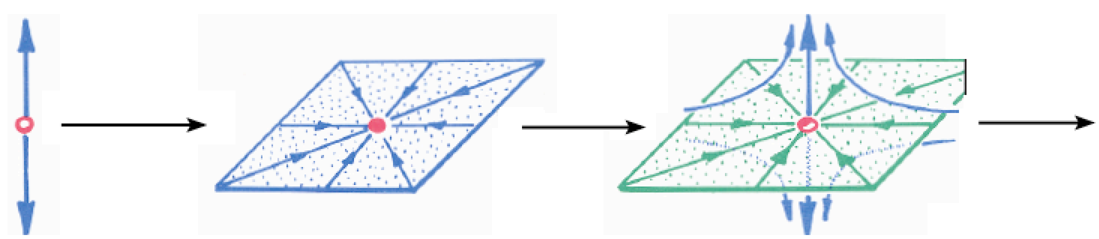


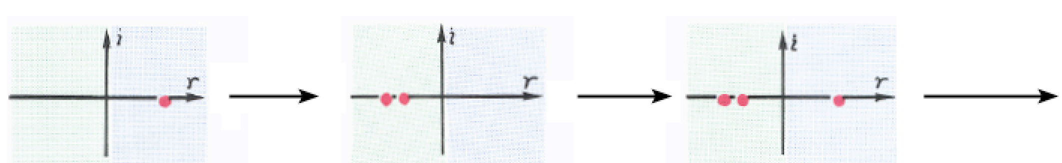
Catastrophe Theory for Scientists and Engineers (1996) R. Gilmore

35_classify4D.psd

Summary: Higher dimensional dynamics

Classify the Dynamics: ***1. Find the fixed points.***
2. Linearize near the fixed points.
3. Compute eigenvalues at fixed points.
4. Classify local stability.
5. Classify bifurcations.





36_summary.psd