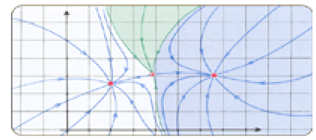
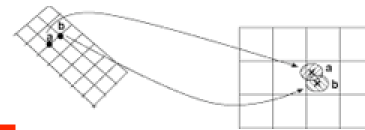
Part 1: Dynamics

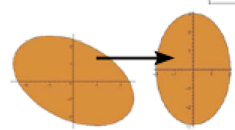
Jan 10 (01) **2-Dimensional flow geometries**. HW1



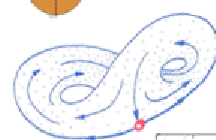
Jan 12 (02) **Discrete dynamics & Mappings**.



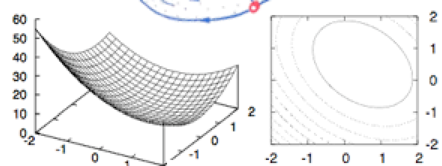
Jan 17 (03) **Diagonalization & eigenvalues**. HW2



Jan 19 (04) **Higher dimensional dynamics & linearization**.



Jan 24 (05) **Stability & Gradient systems**. HW3

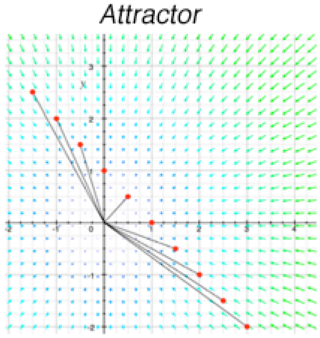


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)
Nonlinear dynamics and chaos, Steven H. Strogatz (1994)
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

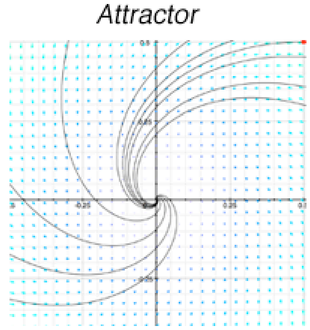
01_Dynamics.psd

Quantify a System's Dynamics:
Step #4: *Classify the Dynamics*

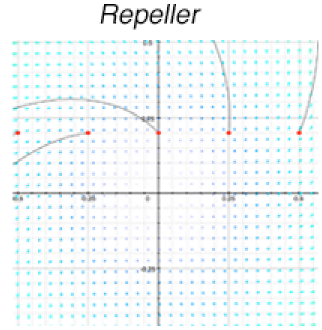
Attractor



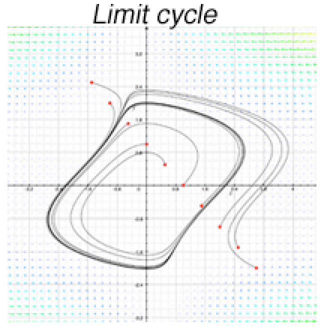
Attractor

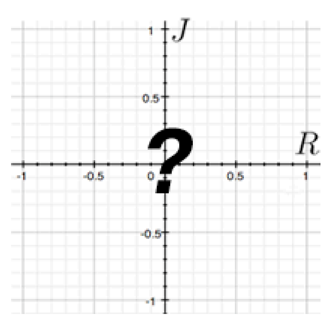


Repeller



Limit cycle

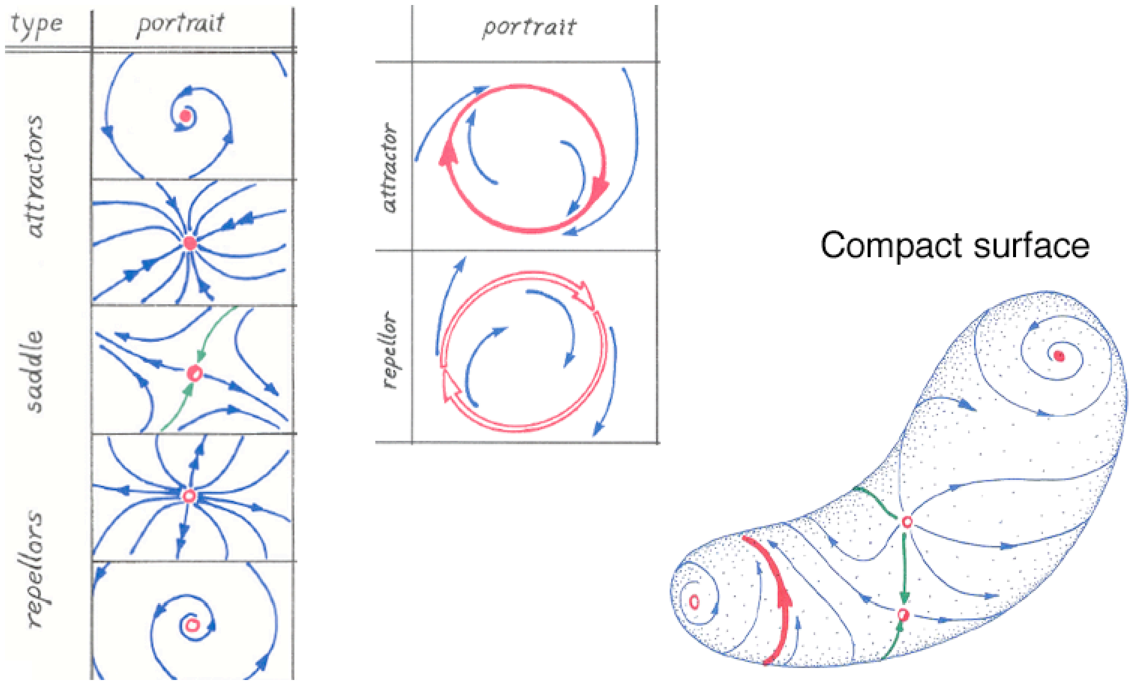




02_classify.psd

The Classification of 2-Dimensional Dynamics

Piexoto's Theorem



Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)

03_Piexoto.psd

The Linearize near the Fixed Points

Given a dynamical system, $$\frac{d}{dt}\vec{x} = f(\vec{x})$$

The fixed points are all points that satisfy the condition,

$$f(\vec{x}_0) = 0$$

Taylor expand near the fixed points:

$$f(\vec{x}) = f(\vec{x}_0) + (\vec{x} - \vec{x}_0) f'(\vec{x})|_{\vec{x}=\vec{x}_0} + \frac{1}{2}(\vec{x} - \vec{x}_0)^2 f''(\vec{x})|_{\vec{x}=\vec{x}_0} + \cdots$$

In the 2-dimensional case, $\vec{x} = (x, y)$

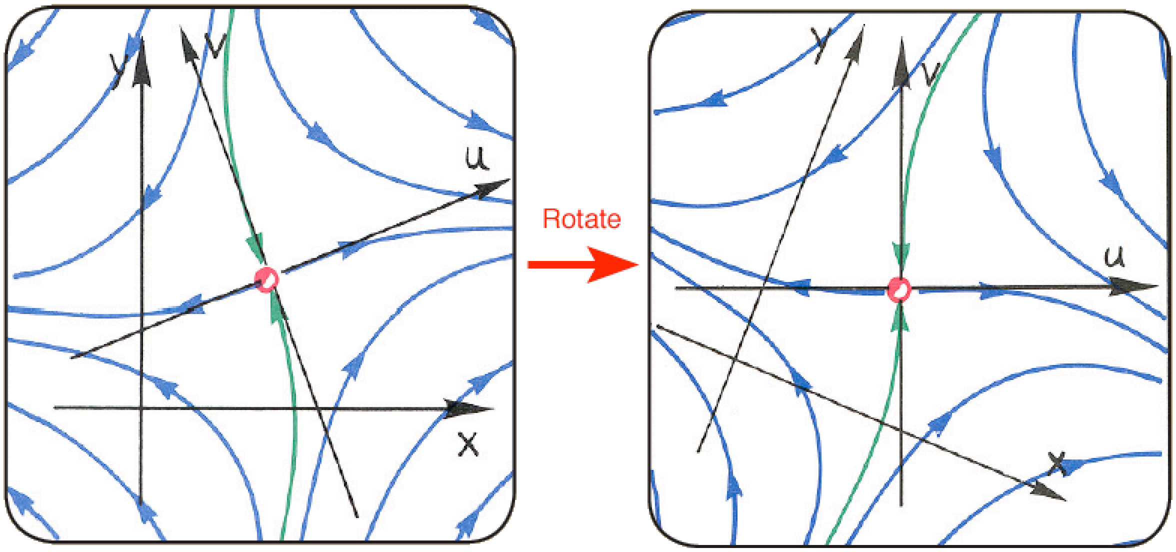
$$\begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} \approx \begin{bmatrix} \frac{df_x}{dx} & \frac{df_x}{dy} \\ \frac{df_y}{dx} & \frac{df_y}{dy} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Find eigenvalues of Jacobian

Why do we expand? "Because it's what we do." -Per Salomonson

04_linearize.psd

Geometrically: Eigenvectors for Orthonormal Basis



05_evals.psd

2-Dimensional Linear Systems

$$\frac{dx}{dt} = a_1 x + a_2 y + b_1 \qquad \frac{dy}{dt} = a_3 x + a_4 y + b_2$$

- Can be written as:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\frac{d\vec{X}}{dt} = \overleftrightarrow{A}\vec{X} + \vec{B}$$

Slide01.png

Equilibrium: Fixed Points

- Equilibrium points occur when the temporal derivative is 0, which defines equilibrium solutions $\vec{X}_{eq}$

$$\frac{d\vec{X}}{dt} = \overleftrightarrow{A}\vec{X} + \vec{B} = 0 \longrightarrow \vec{X}_{eq} = -\overleftrightarrow{A}^{-1}\vec{B}$$

- A *trajectory* is the time course of the system given a particular set of initial conditions
- We can characterize a system by the behavior of its trajectories in the vicinity of the equilibrium points

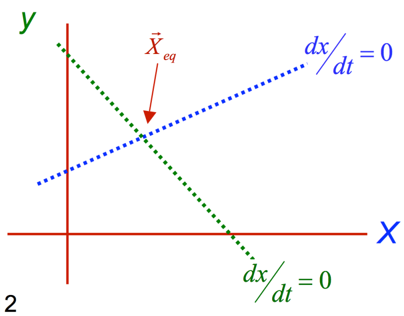
Slide02.png

Stability and state space

- We can plot trajectories in *state space* (also called the *phase plane*) in which the variables of our equations define the axis
- Then, the plots of *dx/dt=0* and *dy/dt=0* are called *nullclines*, and their intersection point represents the equilibrium state of the system

$$\frac{dx}{dt} = a_1 x + a_2 y + b_1$$

$$\frac{dy}{dt} = a_3 x + a_4 y + b_2$$



Slide03.png

Stability and state space (cont.)

- The equilibrium point is *asymptotically stable* if all trajectories starting within a region containing the equilibrium point decay exponentially towards that point
- The equilibrium point is *unstable* if at least one trajectory beginning in a region containing the point leaves the region permanently
- The equilibrium is (neutrally) *stable* if trajectories remain nearby
- The behavior of trajectories can be determined by the eigenvalues of the system

Slide04.png

Linear (Local) Stability

- The behavior of trajectories can be determined by the eigenvalues of the system
- We can transform the system steady state to the origin without changing the dynamics by setting

$$\vec{X}' = \vec{X} - \vec{X}_{eq}$$

- So that $$\frac{d\vec{X}'}{dt} = \overleftrightarrow{A}\vec{X}'$$

$$\frac{d\vec{X}}{dt} = \overleftrightarrow{A}\vec{X} + \vec{B}$$

Slide05.png

Linear Solution in 2-Dimensions

- Now, substitute a vector of exponentials for X with arbitrary (to be determined) coefficients c and d:

$$\vec{X}' = \begin{pmatrix} ce^{\lambda t} \\ de^{\lambda t} \end{pmatrix} = \vec{v}e^{\lambda t}$$

The λ's are the eigenvalues of the system, and the v's are the eigenvectors.

- So,

$$\frac{d\vec{X}'}{dt} = \underline{\lambda\vec{X}' = \overleftrightarrow{A}\vec{X}'} \longrightarrow \left\{\overleftrightarrow{A} - \lambda\overleftrightarrow{I}\right\}\vec{X}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Slide06.png

Characteristic Equation

$$\left\{\overleftrightarrow{A} - \lambda\overleftrightarrow{I}\right\}\vec{X}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution only if $\left\{\overleftrightarrow{A} - \lambda\overleftrightarrow{I}\right\}$

does not have an inverse – which means the determinant vanishes

$$\left|\overleftrightarrow{A} - \lambda\overleftrightarrow{I}\right| = 0$$

The determinant is simply a quadratic polynomial which is the *characteristic equation* of the system

$$\begin{pmatrix} a_1 - \lambda & a_2 \\ a_3 & a_4 - \lambda \end{pmatrix} = 0 \qquad \textit{remember this?}\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Slide07.png

Eigenvalues and Eigenvectors

The solutions of the characteristic equation are called *eigenvalues* of *A*

If the eigenvalues are not equal ($\lambda_1 \neq \lambda_2$) then the solution of our original system

$$\frac{d\vec{X}}{dt} = \overleftrightarrow{A}\vec{X} + \vec{B}$$

is:

$$\vec{X} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ d_1 e^{\lambda_1 t} \end{pmatrix} + \begin{pmatrix} c_2 e^{\lambda_2 t} \\ d_2 e^{\lambda_2 t} \end{pmatrix} + \vec{X}_{eq}$$

So, we only need to determine the *c's* and *d's* (the eigenvectors) to determine the solution for the system of equations

Slide08.png

Solution in New Coordinates

To find the solution for X (i.e. find the *c's* and *d's*), we substitute in our eigenvalue(s)

$$\lambda\vec{X}' = \overleftrightarrow{A}\vec{X}' \qquad \lambda_1 \begin{pmatrix} c_1 e^{\lambda_1 t} \\ d_1 e^{\lambda_1 t} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ d_1 e^{\lambda_1 t} \end{pmatrix}$$

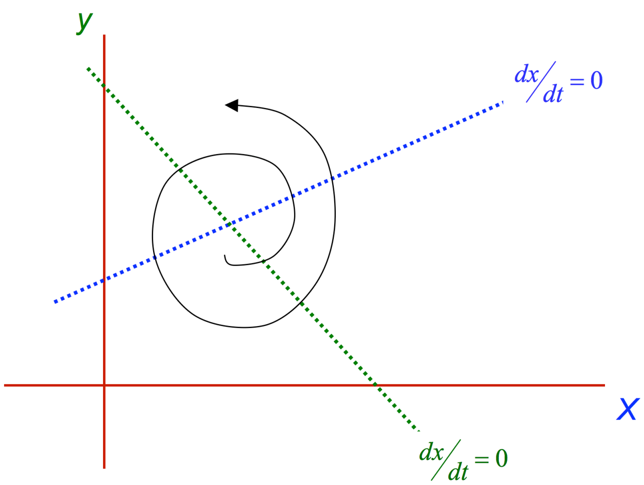
$$\lambda_2 \begin{pmatrix} c_2 e^{\lambda_2 t} \\ d_2 e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} c_2 e^{\lambda_2 t} \\ d_2 e^{\lambda_2 t} \end{pmatrix}$$

Note: we must know the initial conditions to fully determine the *c's* and *d's*

Slide09.png

Stability of Spirals

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Eigenvalues are a complex conjugate pair: equilibrium point is a spiral point.

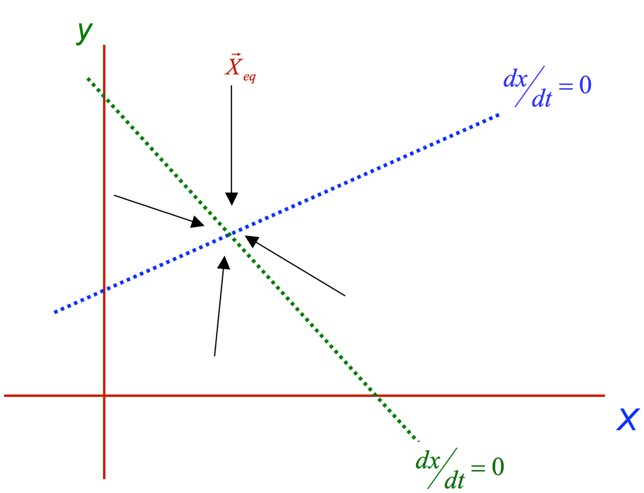
If the real part of the eigenvalues are negative, the point is asymptotically stable

Otherwise, it's unstable

Slide10.png

Stable Fixed Points are Sinks

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Eigenvalues are both real and have the same sign: equilibrium point is a node.

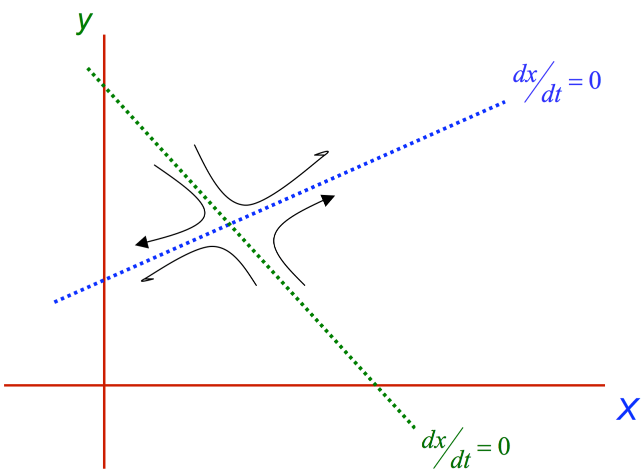
If the eigenvalues are negative, the point is asymptotically stable

Otherwise, it's unstable

Slide11.png

Saddle Points are Stable & Unstable

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



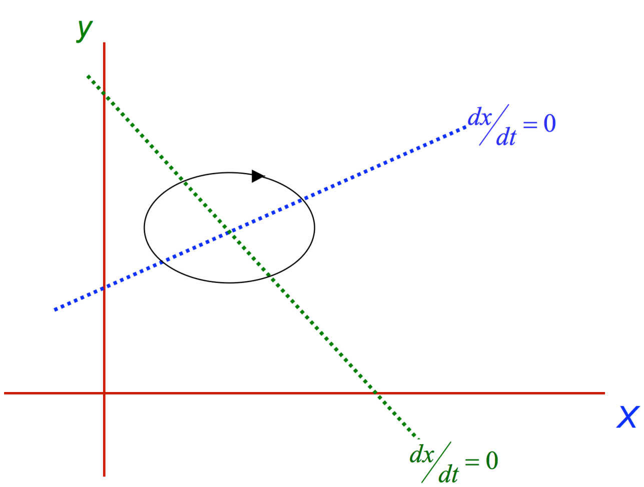
Eigenvalues are both real and have different signs: equilibrium point is a saddle point.

Saddle points are always unstable

Slide12.png

Purely Imaginary Eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Eigenvalues are purely imaginary: equilibrium point is a center.

Centers are neutrally stable, and the trajectory around the equilibrium point will be strictly periodic oscillations

Slide13.png

Non-linear Systems

However, for a general system

$$\frac{du}{dt} = F(u,w) \qquad \frac{dw}{dt} = G(u,w)$$

We can learn a lot about the equilibrium of the system by studying the stability of the steady states (i.e. the temporal equilibrium points):

$$\frac{du}{dt} = 0 \qquad \frac{dw}{dt} = 0$$

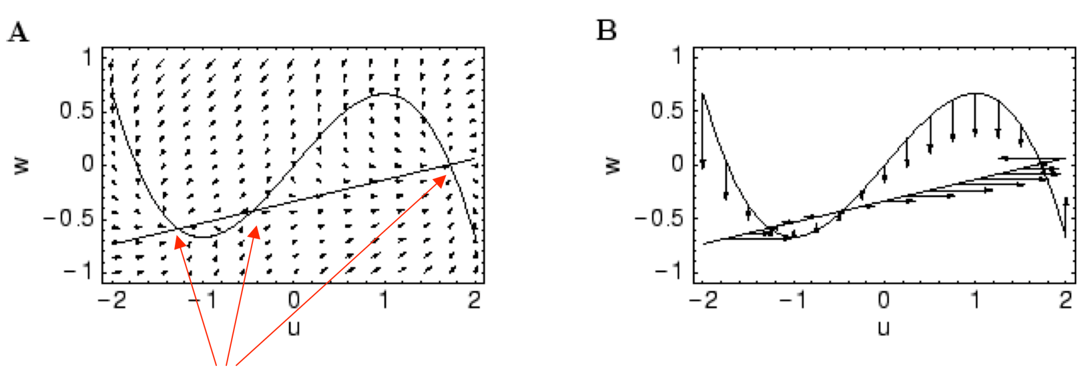
Slide14.png

Non-linear Systems

Again, we do Phase Plane analysis

$\frac{du}{dt} = 0$ u-nullcline

$\frac{dw}{dt} = 0$ w-nullcline



fixed points = equilibrium points

Slide15.png

Non-linear Systems

- We can solve for equilibrium points, but in this case we have non-linear functions, so how do we determine the eigenvalues?

…use the linear terms of the Taylor series expansion around the equilibrium points

$$\frac{du}{dt} = F(u,w) \qquad \frac{dw}{dt} = G(u,w)$$

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial F}{\partial u}\right|_{eq} & \left.\frac{\partial F}{\partial w}\right|_{eq} \\ \left.\frac{\partial G}{\partial u}\right|_{eq} & \left.\frac{\partial G}{\partial w}\right|_{eq} \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

Matrix of first derivatives: Jacobian matrix

Slide16.png

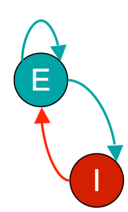
Example

- Consider a very simple Wilson-Cowan system of 2 neuron populations:

$$\tau \frac{dE(x)}{dt} = -E(x) + g_E\left[I^{ext} + w_{EE}E(x) - w_{IE}I(x)\right]$$

$$\tau \frac{dI(x)}{dt} = -I(x) + g_I\left[w_{EI}E(x)\right]$$

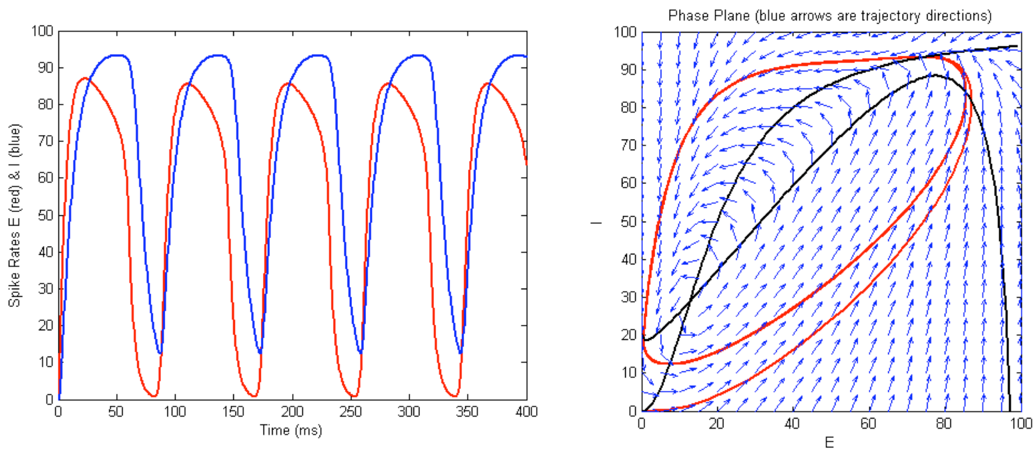
$$g(P) = \begin{cases} \dfrac{100P^2}{30^2 + P^2} & for \;\; P \geq 0 \\ 0 & for \;\; P < 0 \end{cases}$$



Slide17.png

Example (cont.)

$I_{ext} = 20$



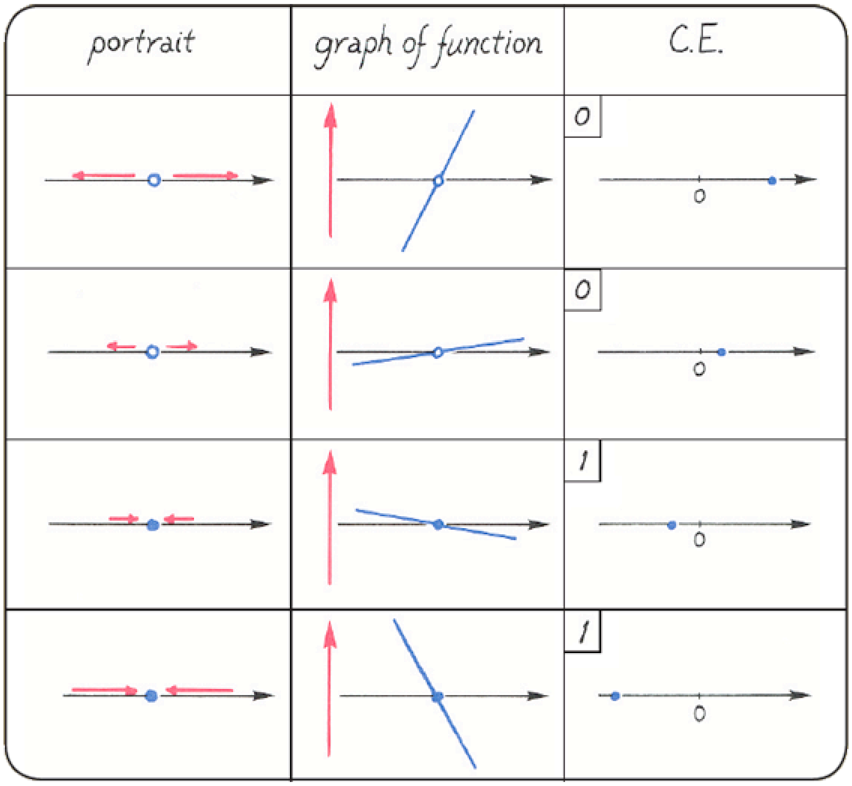
Slide18.png

Limit cycles

- An oscillatory trajectory is a *limit cycle* if all trajectories within a small region enclosing the oscillatory trajectory are spirals
  - If neighboring trajectories spiral towards the oscillatory trajectory, then the limit cycle is asymptotically stable
  - If they spiral away, the limit cycle is unstable
- Poincaré-Bendixon theorem:
  - Suppose there is an annular region that contains no equilibrium points and for which all trajectories that cross the boundary of the annulus enter it
  - Then, the annulus must contain at least one asymptotically stable limit cycle

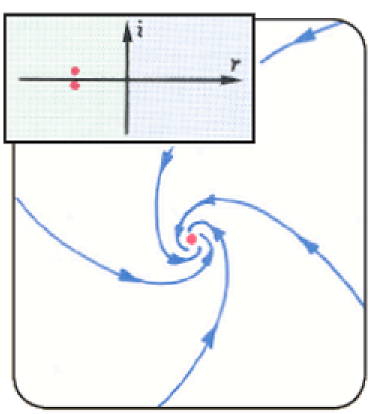
Slide19.png

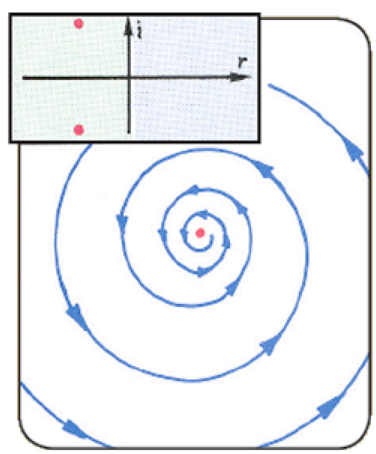
One-Dimension
Eigenvalues (Characteristic Exponents) Determine Source/Sink



06_1Dim.psd

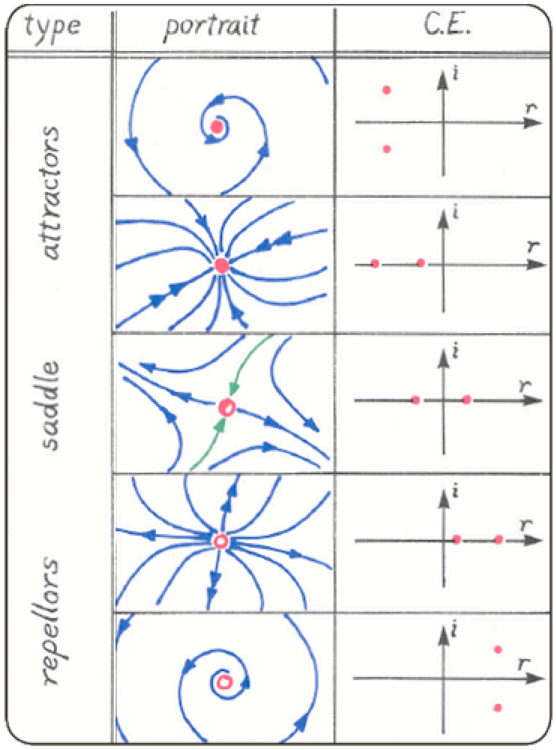
Two-Dimensions
Imaginary Parts of Eigenvalues Determine Spirals





07_2Dim.psd

Two-Dimensions
Eigenvalues Determine the Geometry of Flows

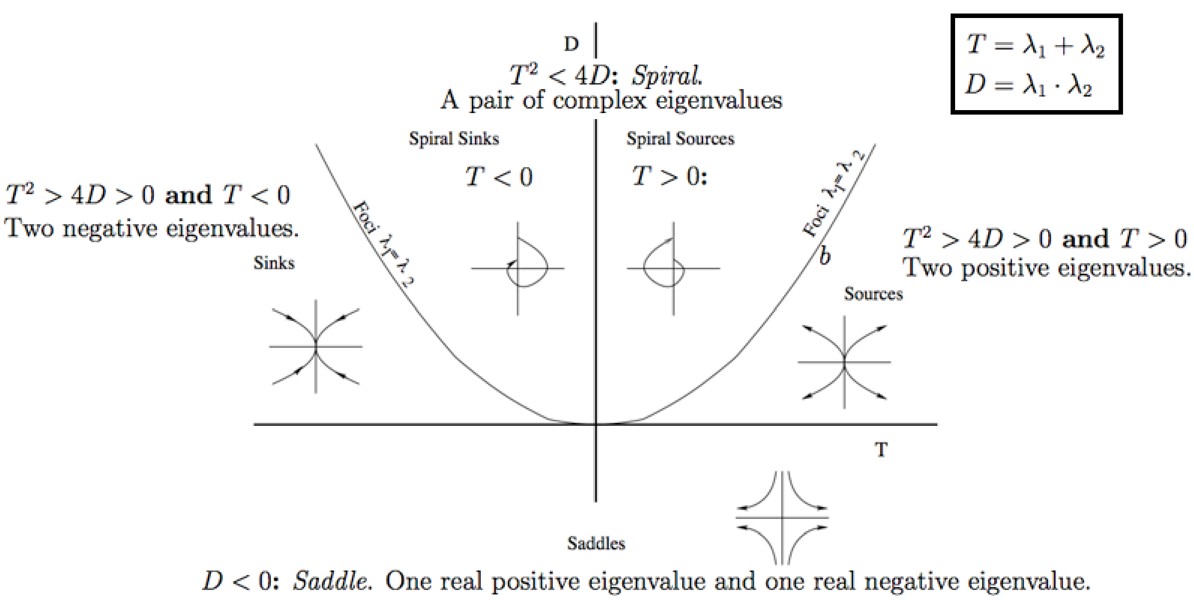


08_peixoto.psd

Great Graph of 2-d Linear Systems

general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with trace $T = a + d$ and determinant $D = ad - cb$:

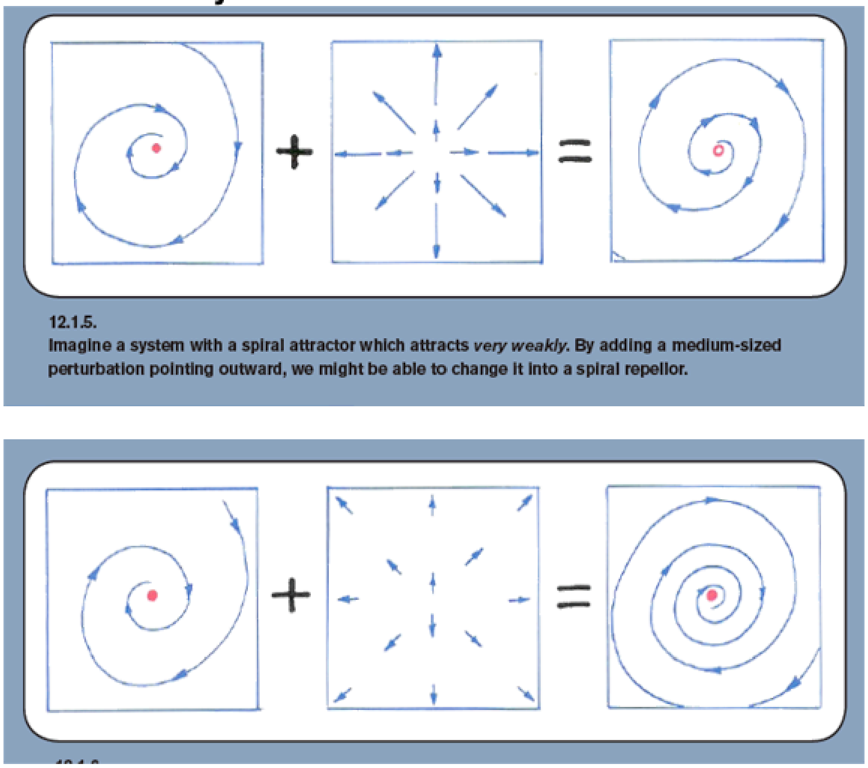
$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - T\lambda + D = 0, \text{ the solutions of which are: } \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$



Fraser's notes (2004)

08b_greatGraph.psd

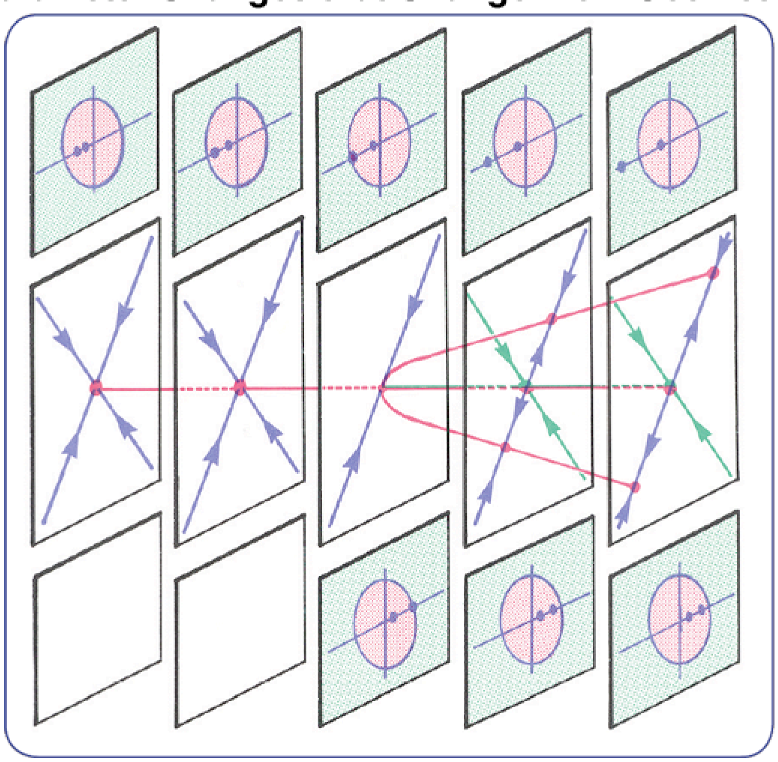
Structural Stability
Flow Geometry is Not Sensitive to Small Perturbations



12.1.5.
Imagine a system with a spiral attractor which attracts *very weakly*. By adding a medium-sized perturbation pointing outward, we might be able to change it into a spiral repellor.

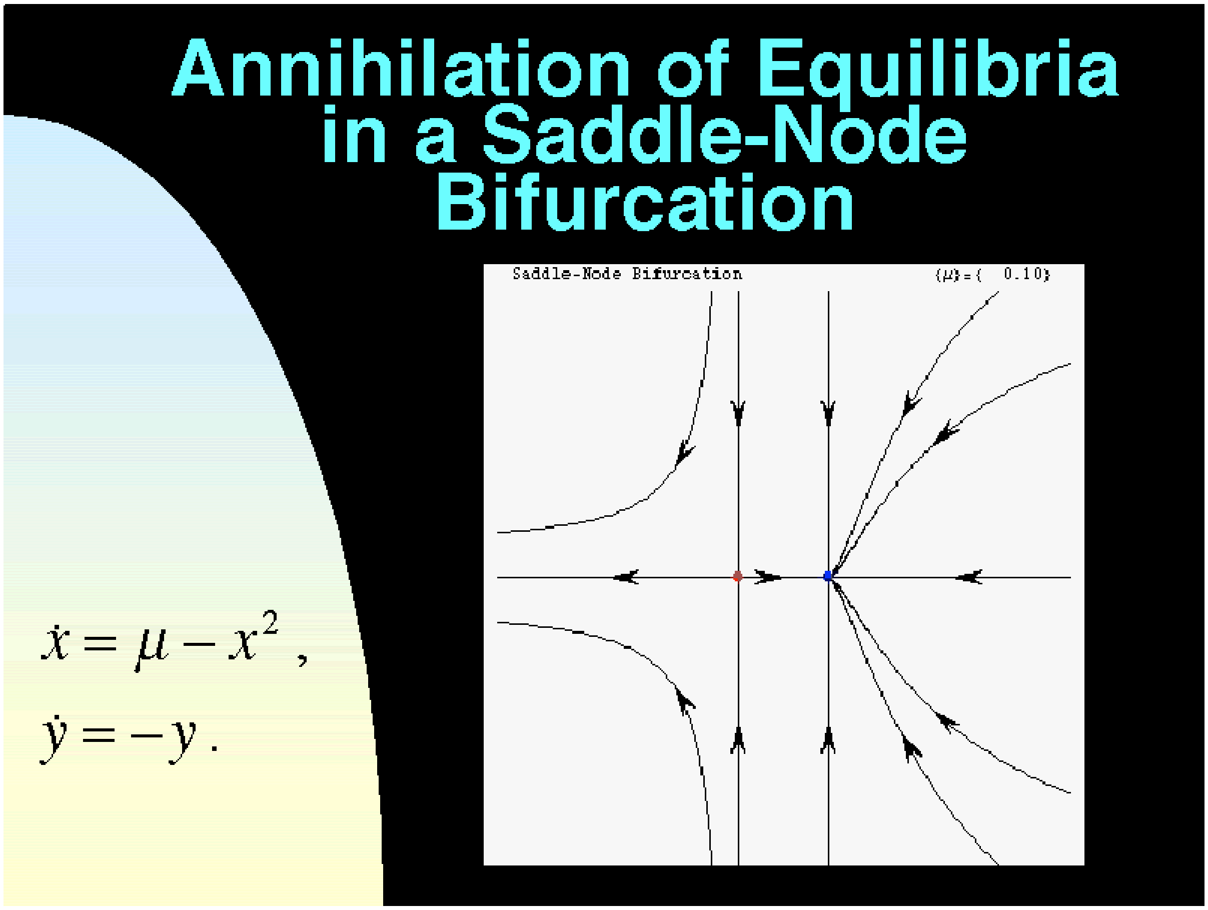
09_StructStabil.psd

Bifurcations
Parameter Changes that Change Flow Geometry

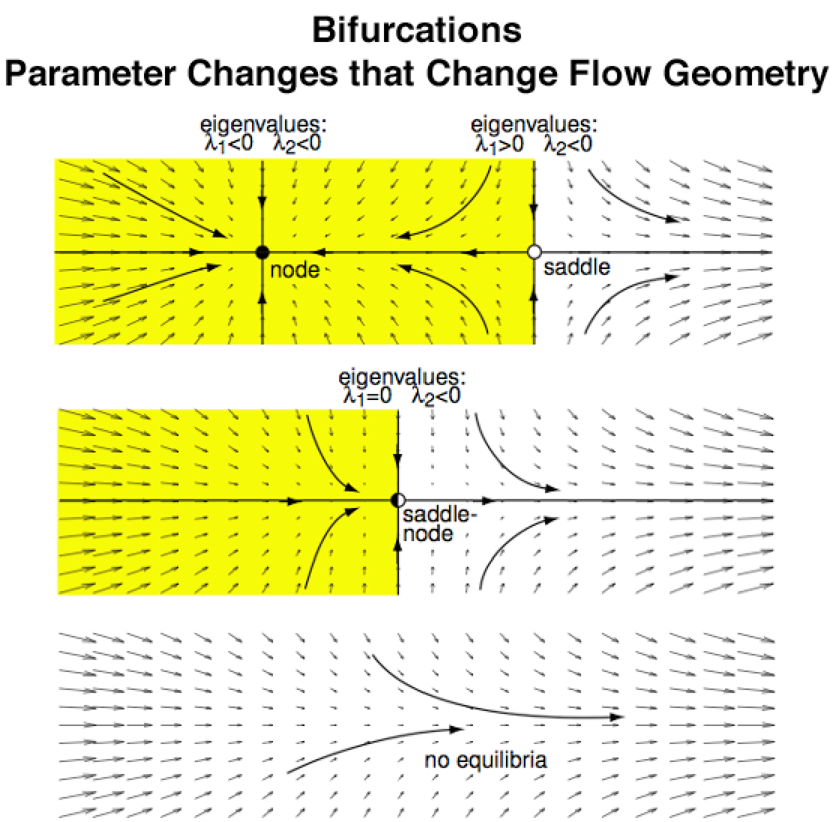


Saddle-node bifurcation

10_bifur.psd

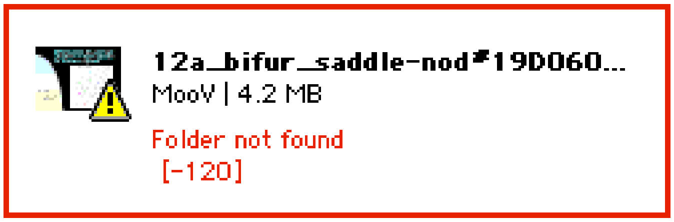


11a_bifur_saddle-node.mov

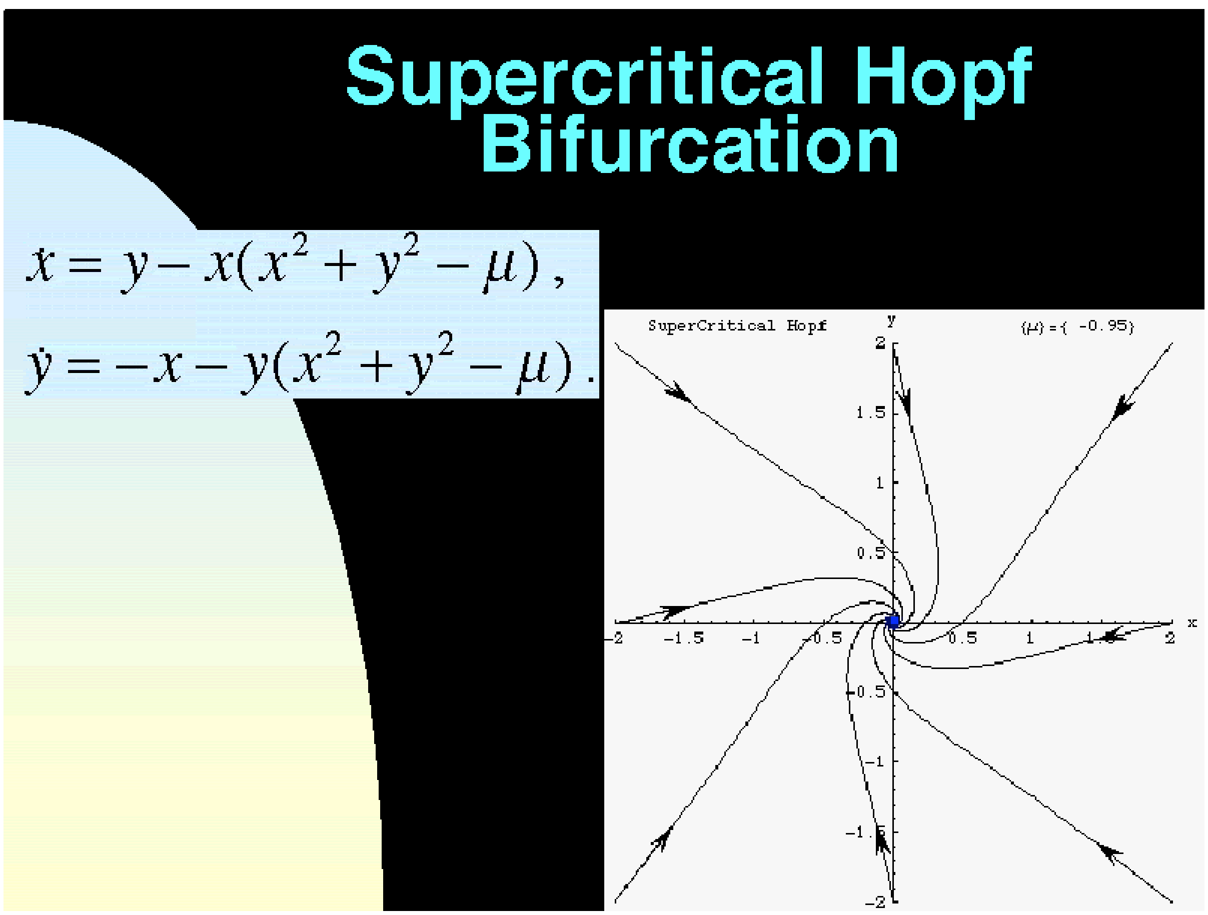


Saddle-node bifurcation

11b_saddleNode.psd

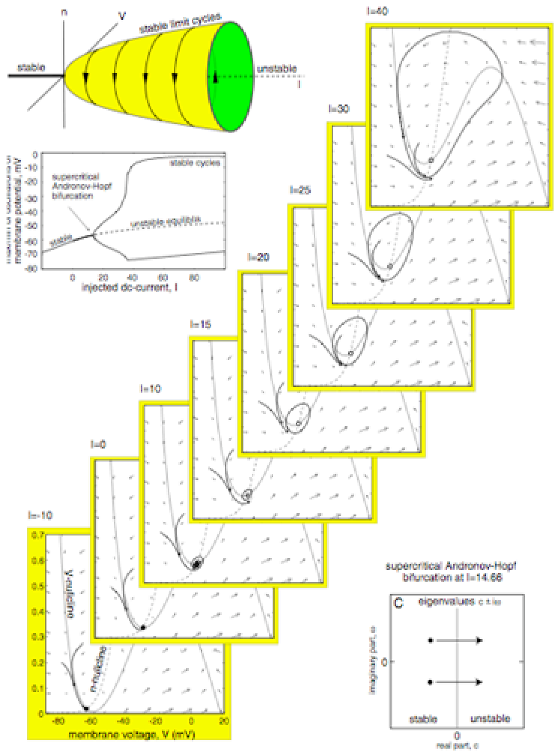


12a_bifur_saddle-nod#19D060.mov



13a_bifur_hopf.mov

Bifurcations
Parameter Changes that Change Flow Geometry



Hopf bifurcation

13b_Hopf.psd

Summary: Eigenvalues Determine Flow Geometries

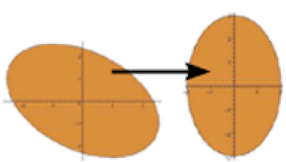
Classify the Dynamics: ***1. Find the fixed points.***

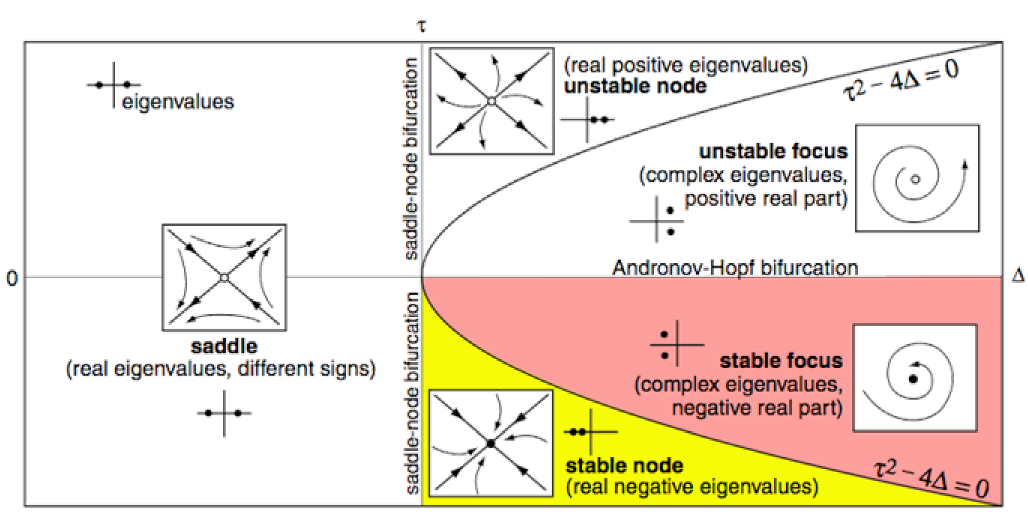
2. Linearize near the fixed points.

3. Compute eigenvalues at fixed points.

4. Classify local stability.

5. Classify bifurcations.





15_summary.psd