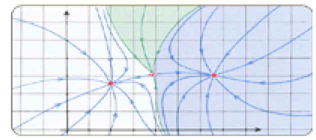
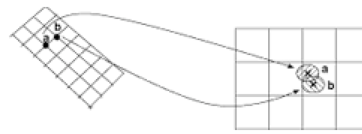
Part 1: Dynamics

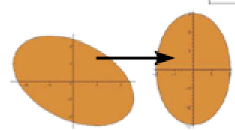
Jan 10 (01) **2-Dimensional flow geometries**. HW1



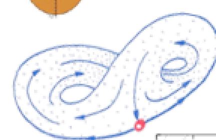
Jan 12 (02) **Discrete dynamics & Mappings**.



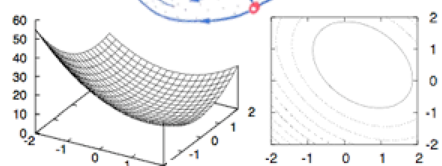
Jan 17 (03) **Diagonalization & eigenvalues**. HW2



Jan 19 (04) **Higher dimensional dynamics & linearization**.



Jan 24 (05) **Stability & Gradient systems**. HW3

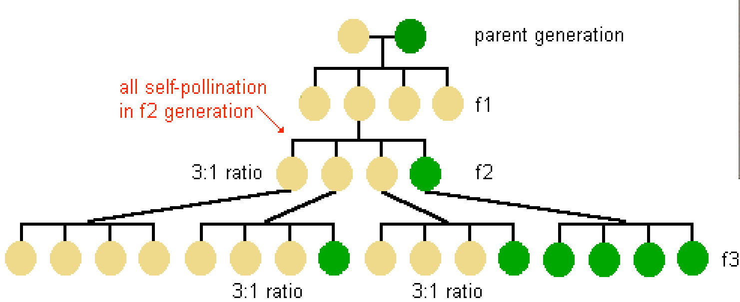


Dynamics: The Geometry of Behavior, Ralph Abraham and Chris Shaw (2005)
Nonlinear dynamics and chaos, Steven H. Strogatz (1994)
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

01_Dynamics.psd

Population Dynamics

Step 1. Draw a distinction
Multi-generational population



Step 2. Specify an example
Mendel's Peas

How many survive with each generation?

Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

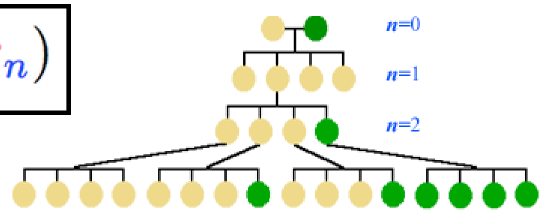
02_PopulDyn.psd

Population Dynamics

Step 3. Quantify the salient features.

How many survive with each generation?

Discrete Problem: $$x_{n+1} = f(x_n)$$



Step 4. Classify the dynamics.

Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

03_PopulDyn.psd

Population Dynamics: Unlimited Resources

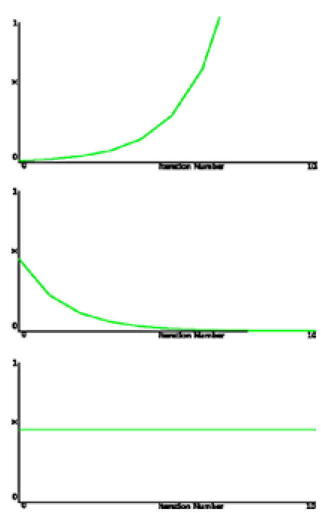
$$x_{n+1} = ax_n$$

Exponential Growth or Decay

If $|a| > 1$, then exponential growth.

If $|a| < 1$, then exponential decay.

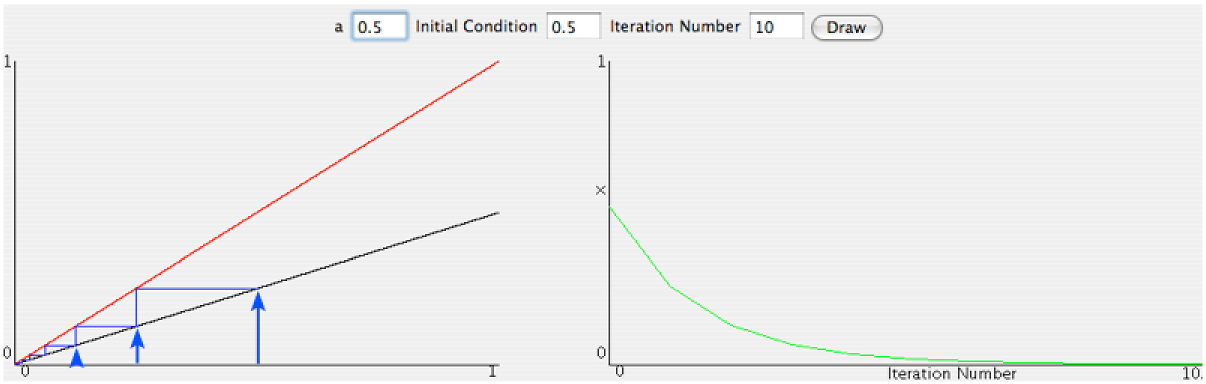
If $a = 1$, then perfect replenishment.



Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

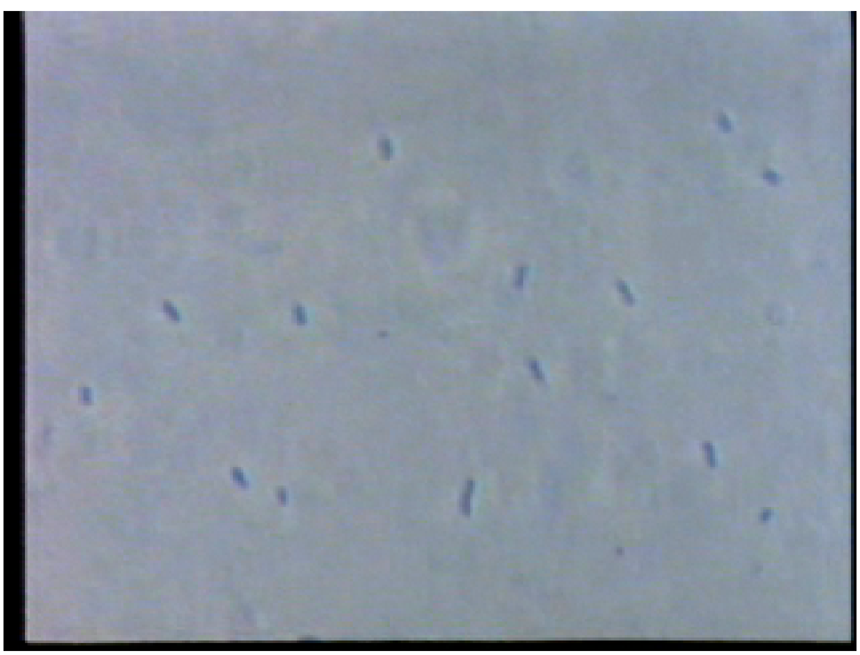
04_unlimited.psd

Linear, Recursive Mapping



$$x_{n+1} = ax_n$$

05_simLinear.psd



06a_bacterial_division.mov

Population Dynamics: Bacterial Growth

Step 1. Draw a distinction
Multi-generational population

Step 2. Specify an example
Fast Reproducing E. coli

Continuum Exponential Growth or Decay

$$\frac{dx}{dt} = ax$$

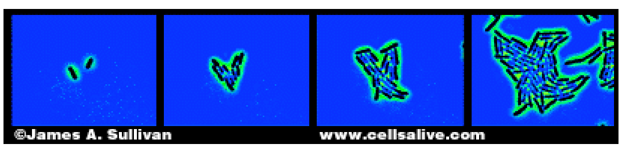
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

06b_bacteria.psd

Population Dynamics: Limited Bacterial Growth

Step 1. Draw a distinction.
Bacterial population + resources

Step 2. Specify an example.
Fast Reproducing E. coli



Limited Resources = Limited Growth

$$\frac{dx}{dt} = ax(1 - x)$$

Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

07_limited.psd

Digression: Solving differential equations

Logistic equation: $\frac{dx}{dt} = ax(1-x)$

$$\frac{dx}{x(1-x)} = a\,dt$$

Use partial fractions by solving for A & B: $\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$

$$\begin{aligned}\left[\frac{1}{x} + \frac{1}{(1-x)}\right] dx &= a\,dt \\ \ln(x) - \ln(1-x) &= at + c \\ \frac{x}{1-x} &= e^{at+c}\end{aligned}$$

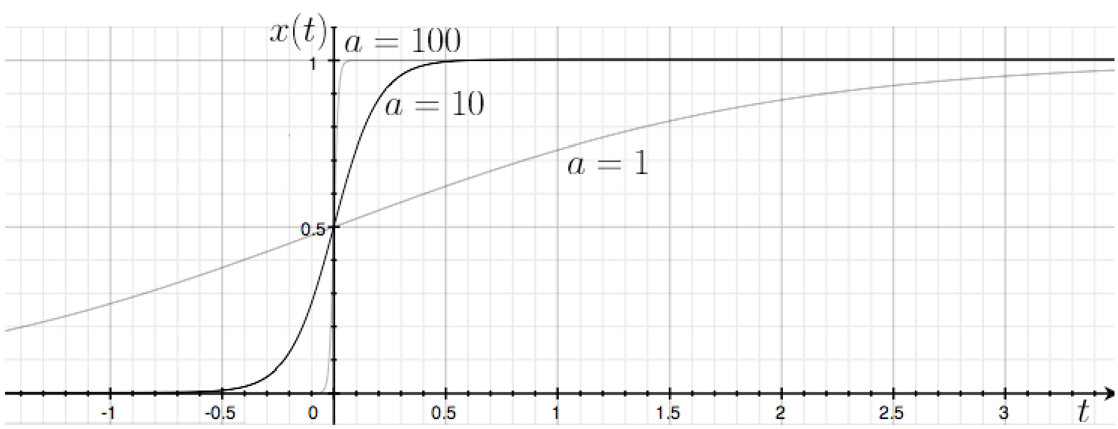
Sigmoid solution (saturation): $\boxed{x(t) = \frac{1}{1+e^{-at+c}}}$

08_solve.psd

Population Dynamics: Limited Bacterial Growth

$$\boxed{x(t) = \frac{1}{1+e^{-at+c}}}$$

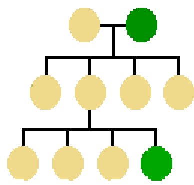
Saturation with Growth Rate Parameter:



09_saturation.psd

Return to the Discrete Case: Annual Generations

Step 1. Draw a distinction
Multi-generational + limited resources



Step 2. Specify an example
Mendel's Peas

What are the new saturation dynamics?

Discrete logistic equation:

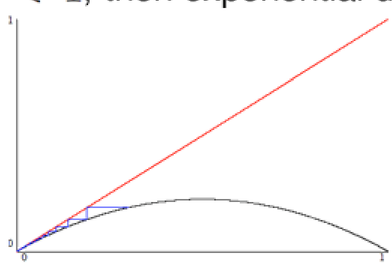
$$x_{n+1} = a x_n (1 - x_n)$$

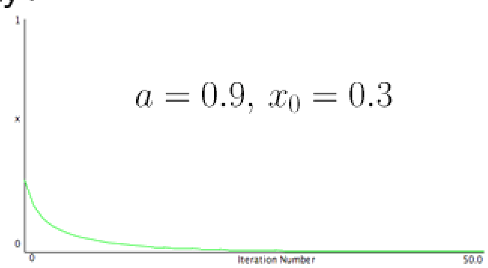
10_limitedPeas.psd

Discrete Population Dynamics: Limited Resources

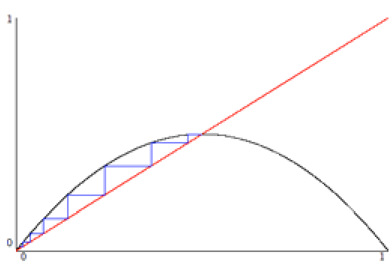
$$x_{n+1} = a x_n (1 - x_n)$$

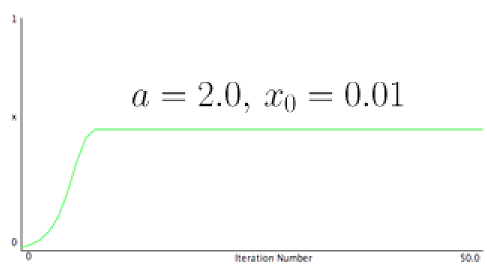
If $|a| < 1$, then exponential decay?





If $|a| > 1$, then growth until saturation?





11_limitedDyn.psd

Fixed Points of a Mapping

$$\begin{aligned} x_{n+1} &= f_a(x_n) \\ &= a x_n (1 - x_n) \end{aligned}$$

What are the fixed points, $\bar{x} = f_a(\bar{x})$?

Fixed Point Theorem. *Suppose that the map $f_a(x)$ has a fixed point at $\bar{x}$. The the fixed point is stable if*

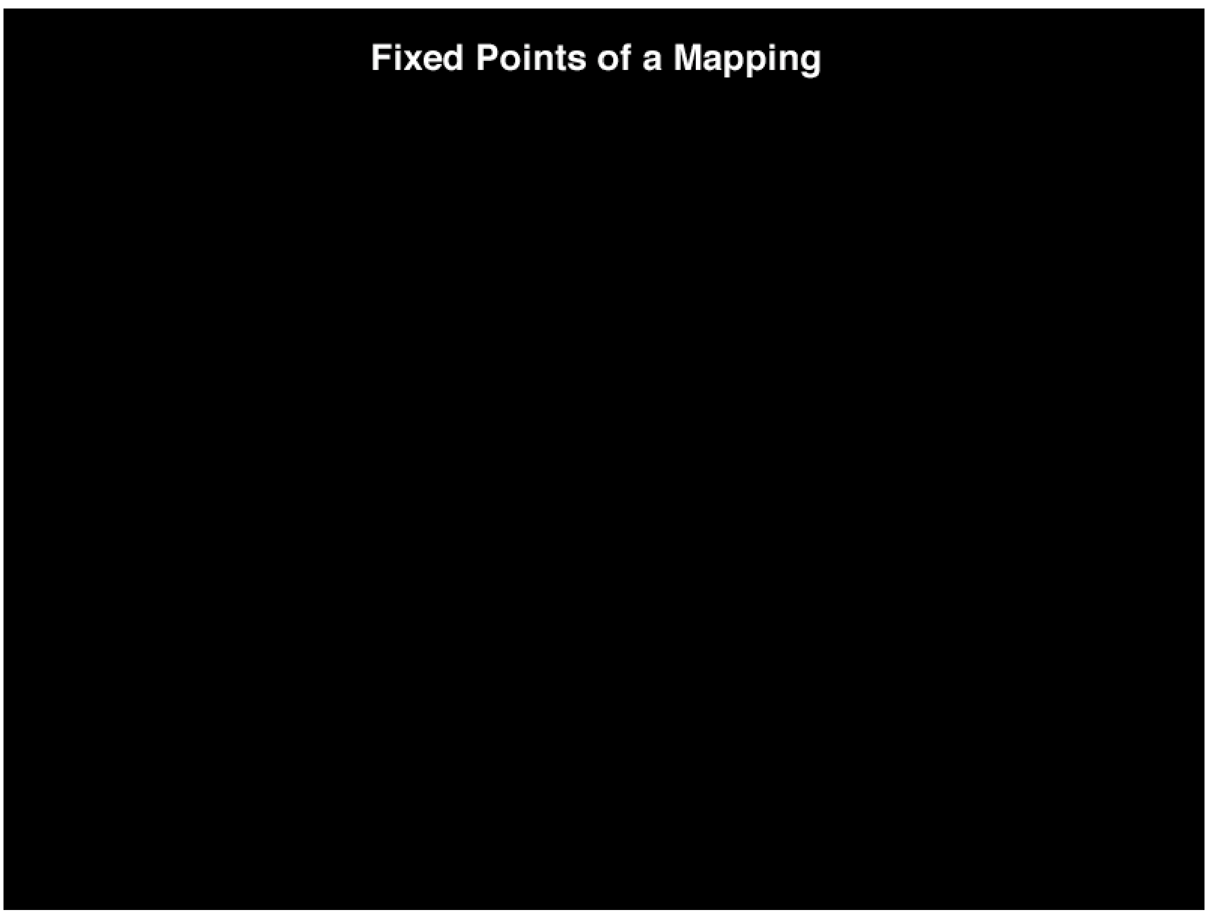
$$\left| \frac{d}{dx} f_a(\hat{x}) \right| < 1,$$

and it is unstable if

$$\left| \frac{d}{dx} f_a(\hat{x}) \right| > 1.$$

Nonlinear Dynamics and Chaos, Stephen Lynch(2004)

11b_fixedPt.psd

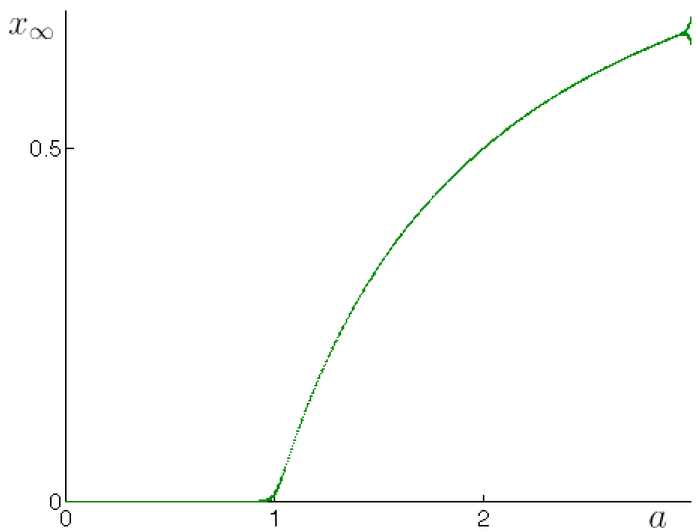


11c_blackboard.psd

Discrete Population Dynamics: Limited Resources

$$x_{n+1} = ax_n(1 - x_n)$$

Bifurcation Diagram for the Fixed Point (100 iterations):



12_limitedDynBifur.psd

Fixed Points of a Mapping

$$\begin{aligned} x_{n+2} &= f_a(f_a(x_n)) \\ &= a^2 x_n(1 - x_n)(1 - ax_n(1 - x_n)) \end{aligned}$$

What are the fixed points of period two, $x_{n+2} = f_a(f_a(x_n))$?

Fixed Point Theorem. *Suppose that the map $f_a(x)$ has a fixed point at $\bar{x}$. The the fixed point is stable if*

$$\left|\frac{d}{dx} f_a(\hat{x})\right| < 1,$$

and it is unstable if

$$\left|\frac{d}{dx} f_a(\hat{x})\right| > 1.$$

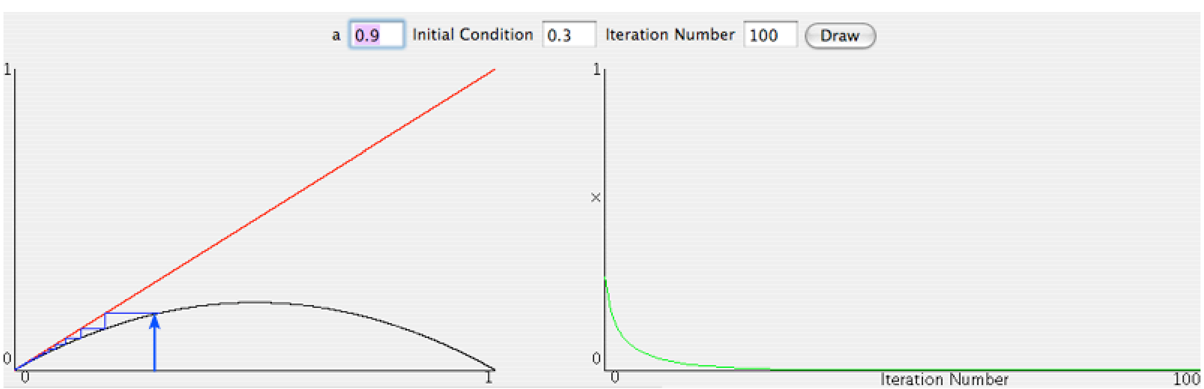
Nonlinear Dynamics and Chaos, Stephen Lynch(2004)

12b_fixedPt.psd

Fixed Points of a Mapping

12c_blackboard.psd

Logistic Mapping



$$x_{n+1} = a x_n (1 - x_n)$$

13_simLogistic.psd

Fast Growth in the Discrete Case: Large Generations

Step 2. Specify an example

Mendel's Peas *Plague of Locusts*

What are the dynamics as the growth parameter becomes large?

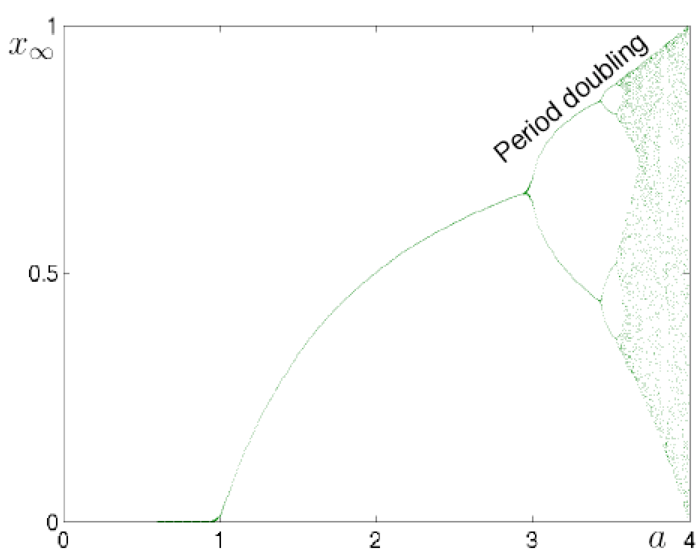
Discrete logistic equation: $$x_{n+1} = ax_n(1 - x_n)$$

14_locusts.psd

Discrete Population Dynamics: Limited Resources

$$x_{n+1} = ax_n(1 - x_n)$$

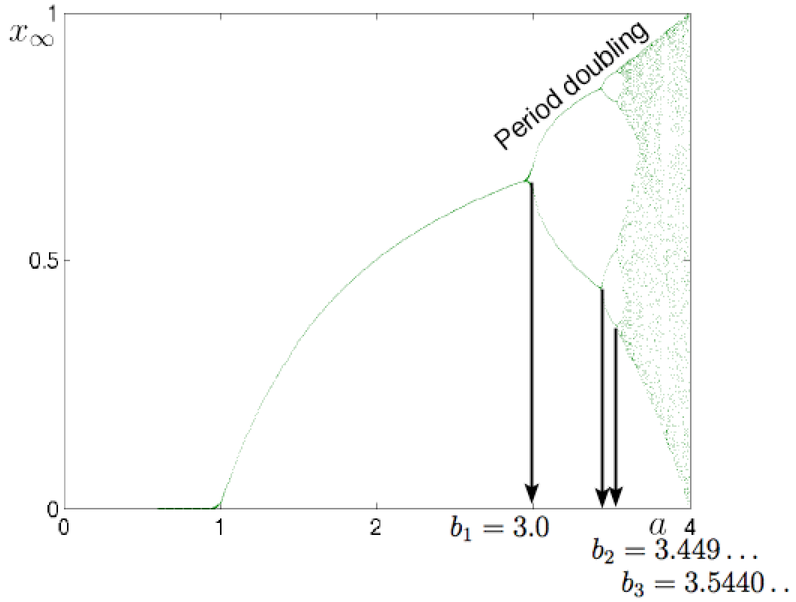
Bifurcation Diagram for the Fixed Point (100 iterations):



Chaos at high growth rates!

15_limitedDynBifur4.psd

Universality: Feigenbaum Number



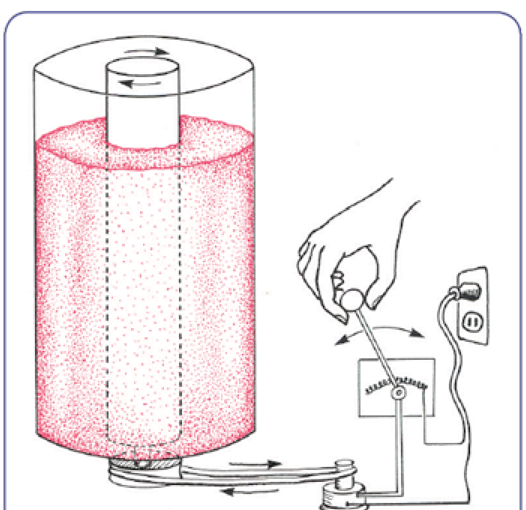
$$d_k = b_{k+1} - b_k$$

$$\delta = \lim_{k\to\infty} \frac{d_k}{d_{k+1}} = 4.669202\ldots$$

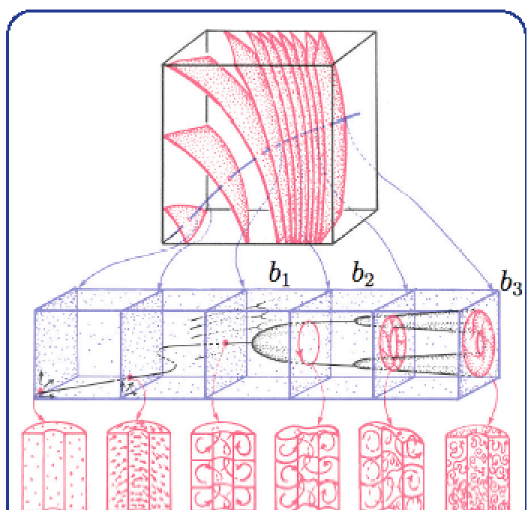
Found in all "period -doubling" cascades to chaos.

15b_feigenbaum.psd

Universality: Feigenbaum Number
Onset of Turbulence



Mallock and Couette motion



Period doubling

$$d_k = b_{k+1} - b_k$$

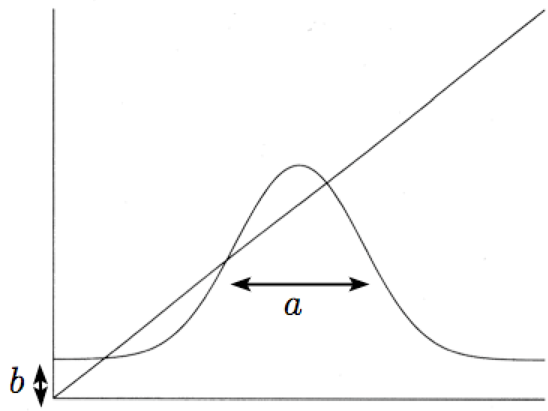
$$\delta = \lim_{k\to\infty} \frac{d_k}{d_{k+1}} = 4.669202\ldots$$

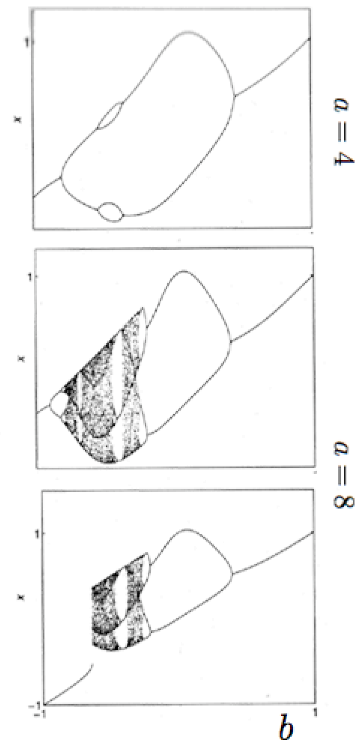
15c_turbulence.psd

Gaussian Map

$$x_{n+1} = f_a(x_n)$$

$$f_{a,b}(x_n) = e^{-ax_n^2} + b$$





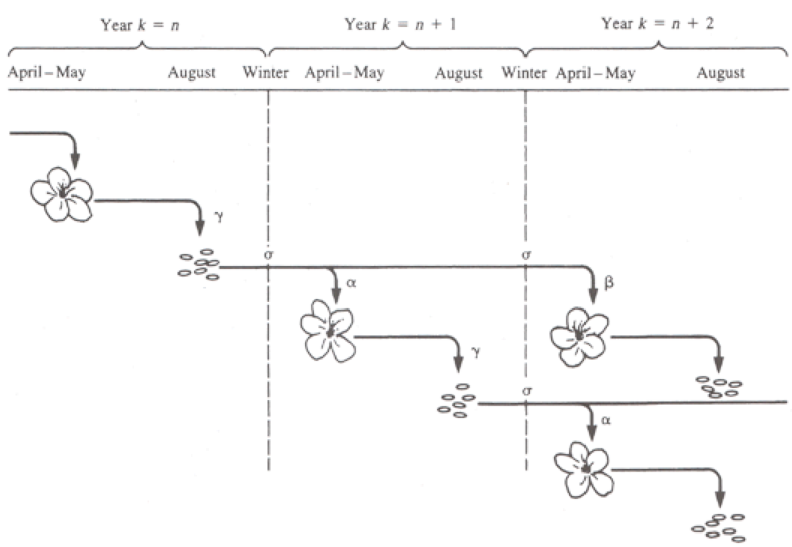
Nonlinear Dynamics and Chaos, Stephen Lynch(2004)

15d_GaussianMap.psd

Second-Order Population Dynamics

Step 1. Draw a distinction
Multi-seasonal population

Step 2. Specify an example
Peas, again...

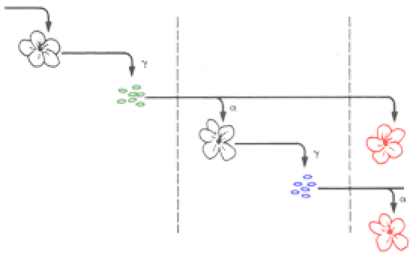


Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

16_2ndSeasonPopul.psd

Second-Order Population Dynamics

Step 3. Quantify the salient features.



Let x_n be the numbers of peas in year n.
Let y_n be the number of plants.
Let α be the fraction of seeds that sprout.
Let γ be the number of peas per plant.

$$\begin{aligned} x_n &= \gamma y_n \\ y_n &= \alpha x_{n-1} + (1-\alpha) x_{n-2} \end{aligned}$$

Eliminate the plant-variable to arrive at a second-order difference equation in peas:

$$\boxed{x_{n+2} = \gamma(\alpha x_{n+1} + (1-\alpha) x_n)}$$

Step 4. Classify the dynamics.

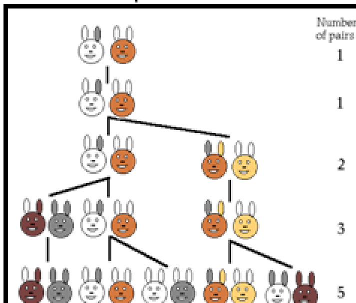
Mathematical Models in Biology, Leah Edelstein-Keshet (1988)

17_2ndSeasonPopul.psd

Fibonacci Sequence

Let $\gamma = 2$ and $\alpha = 0.5$, then we arrive at the Fibonacci sequence:

$$x_{n+2} = x_{n+1} + x_n$$

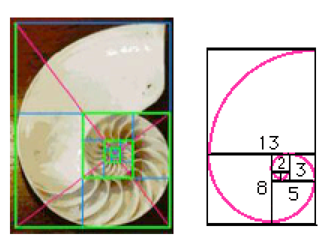


Fibonacci's rabbits (1202)

The Fibonacci sequence can be represented by a 2-dimensional system:

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$\boxed{\vec{x}_{n+1} = \mathbf{A} \vec{x}_n}$$



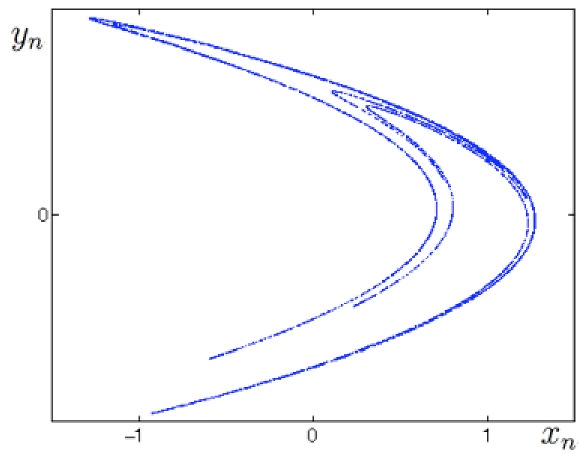
Chambered Nautilus

18_Fibonacci.psd

Hénon Map

In 1976, M.Hénon described a simple map that has a strange attractor ($a = 1.4$ and $b = 0.3$).

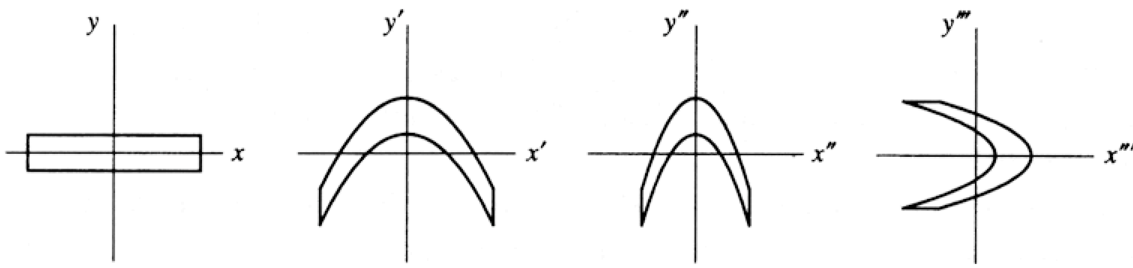
$$
\begin{aligned}
x_{n+1} &= 1 - a x_n^2 + y_n \\
y_{n+1} &= b x_n
\end{aligned}
$$



The Hénon map is widely studied. It is invertible and contracts areas.

20_HenonMap.psd

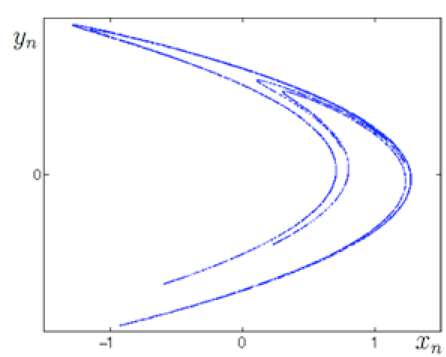
Transformations that Generate the Hénon Map



T': $x' = x$, $y' = 1 + y - ax^2$
T'': $x'' = bx'$, $y'' = y'$
T''': $x''' = y''$, $y''' = x''$

Then the composition yields the Hénon map

$$T = T'''T''T'$$



21_HenonTransf.psd

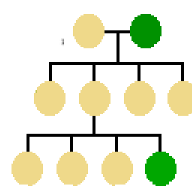
Fixed Points of the Hénon Map

21b_blackboard.psd

Summary: Discrete Mappings

Applications: ***Discrete processes***

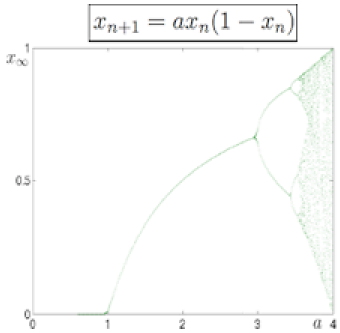
$$x_{n+1} = f_a(x_n)$$



Classify the Dynamics: ***1. Find the fixed points***
2. Determine stability

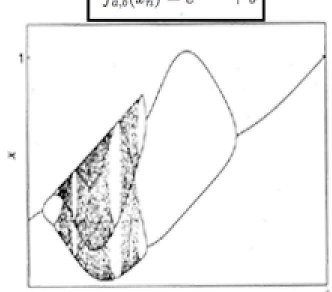
Logistic Map

$$x_{n+1} = ax_n(1-x_n)$$



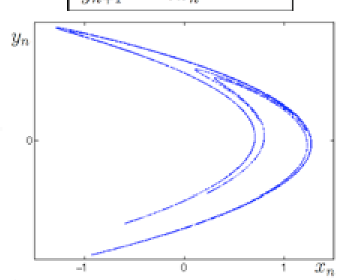
Gaussian Map

$$f_{a,b}(x_n) = e^{-ax_n^2} + b$$



Hénon Map

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n \end{aligned}$$

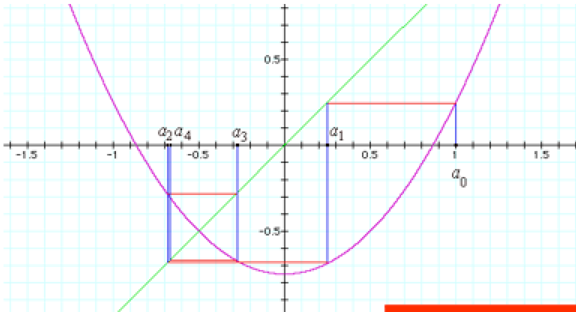


22_summary.psd

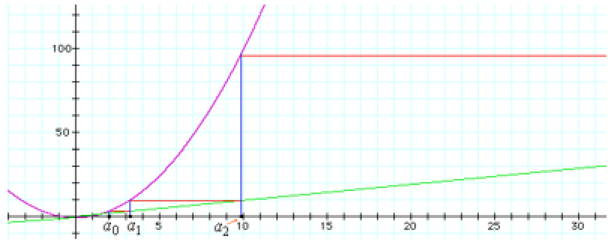
Function Iteration and Julia Sets

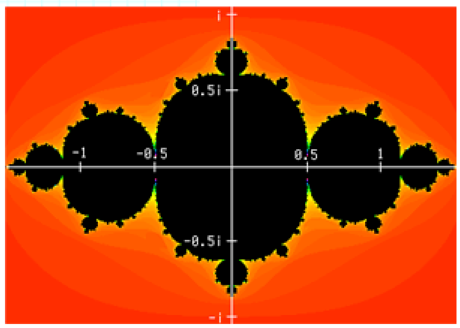
$$z_{n+1} = z_n^2 + c$$

$c = -0.75$



$c = -0.75$





24_julia.psd