SySc 512, Session 18, HW6.1

1) a)

$$\mathbb{E}\tau = \int_0^\infty t\, P_\tau(t)\,dt = R\int_0^\infty t\, e^{-Rt}\,dt$$

Since, $\boxed{\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}} \Rightarrow \quad = R\left(\frac{1}{R^2}\right) = \frac{1}{R}$

$$\mathbb{D}\tau = \mathbb{E}\left(\tau - \frac{1}{R}\right)^2 = \mathbb{E}(\tau)^2 - (\mathbb{E}\tau)^2$$

$$= \mathbb{E}(\tau)^2 - \frac{1}{R^2}$$

Since, $\mathbb{E}(\tau^2) = R\int_0^\infty t^2 e^{-Rt}\,dt = R\left[\frac{2}{R^3}\right] = 2\,\frac{1}{R^2}$

$$= 2\,\frac{1}{R^2} - \frac{1}{R^2} = \frac{1}{R^2}$$

b) $P_\tau^3(t) = \frac{(3R)^3}{2} t^2 e^{-3Rt}$

$$\mathbb{E}\tau_3 = \int_0^\infty t\, P_\tau^3(t)\,dt = \frac{(3R)^3}{2}\int_0^\infty t^3 e^{-3Rt}$$

$$= \frac{(3R)^3}{2}\left(\frac{3\cdot 2}{(3R)^4}\right) = \frac{1}{R}$$

[In fact,

$$\mathbb{E}\tau_n = \int_0^\infty t\, P_\tau^n(t)\,dt = \frac{(nR)^n}{(n-1)!}\int_0^\infty t^n e^{-nRt}$$

$$= \frac{\cancel{(nR)^n}}{(n-1)!}\,\frac{n!}{(nR)^{n+1}} = \frac{n}{nR} = \frac{1}{R}$$
]

$$\mathbb{D}\tau_3 = \mathbb{E}(\tau^2) - \frac{1}{R^2}$$

Since $\mathbb{E}(\tau^2) = \frac{(3R)^3}{2}\int_0^\infty t^4 e^{-3Rt} = \frac{(3R)^3}{2}\left(\frac{4\cdot 3\cdot 2}{(3R)^5}\right)$

$$\Rightarrow \quad = \frac{4}{3}\frac{1}{R^2} - \frac{1}{R^2} = \frac{1}{3}\frac{1}{R^2}$$

(w/ $\Gamma(n) = n!$
$\Gamma(\frac{n}{2}+1) = \sqrt{\pi}\frac{n!!}{\sqrt{2^{n+1}}}$

SySc 512 Session 18, HW 6.2

2) a) To "integrate-out" $\hat{x}_2$, complete the square

$$\hat{x}_1^2 - 2\rho\hat{x}_1\hat{x}_2 + \hat{x}_2^2 + \rho^2\hat{x}_1^2 - \rho^2 x_1^2 =$$
$$= \hat{x}_1^2 + (\hat{x}_2 - \rho\hat{x}_1)^2 - \rho^2\hat{x}_1^2$$

$\Rightarrow$

$$P_{\xi_1}(\hat{x}_1) = A\int_{-\infty}^{\infty}\exp\left[\frac{-((1-\rho^2)\hat{x}_1^2 + (\hat{x}_2 - \rho\hat{x}_1)^2}{2(1-\rho^2)}\right]d\hat{x}_2$$

Since, $\frac{\Gamma(\frac{1}{2}(n+1))}{\sqrt{a^{n+1}}} = \int_{-\infty}^{\infty} x^n e^{-ax^2}dx =$ ~~...~~

$$\Rightarrow = A\exp\left[-\frac{1}{2}\hat{x}_1^2\right]\int_{-\infty}^{\infty}\exp\left[\frac{(\hat{x}_2 - \rho\hat{x}_1)^2}{2(1-\rho^2)}\right]d\hat{x}_2\,\sigma_2$$

$$= A\exp\left[-\frac{1}{2}\hat{x}_1^2\right]\left[\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2(1-\rho^2)}}}\right] = \frac{\sqrt{2\pi(1-\rho^2)}\,\sigma_2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2}\hat{x}_1^2}$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma_1}e^{-\frac{1}{2}\hat{x}_1^2}$$

~~$\mathbb{E}\xi_1 = \int_{-\infty}^{\infty}\hat{x}_1 P_{\xi_1}(\hat{x}_1)d\hat{x}_1\sigma_1 = \frac{1}{\sqrt{2\pi}}$~~

$$\mathbb{E}\xi_1 = \int_{-\infty}^{\infty} x_1 P_{\xi_1}(x_1)dx_1 = \frac{1}{\sqrt{2\pi}\,\sigma_1}\int_{-\infty}^{\infty} x_1 e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}dx_1$$

$(x = x_1 - \mu) \rightarrow$
$$= \frac{1}{\sqrt{2\pi}\,\sigma}\cancelto{0}{\int_{-\infty}^{\infty} x e^{-\frac{1}{2}(x/\sigma_1)^2}dx} + \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}\mu e^{-\frac{1}{2}(x/\sigma)^2}dx$$
$$= \mu_1$$

$$\mathbb{D}\xi_1 = \mathbb{E}(\xi_1^2) - (\mathbb{E}\xi_1)^2$$
$$= \int_{-\infty}^{\infty} x_1^2 P_{\xi_1}(x_1)dx_1 - \mu^2$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma_1}\int_{-\infty}^{\infty}(x^2 + \cancelto{0}{2\mu x} + \mu^2)e^{-\frac{1}{2}(x/\sigma_1)^2}dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma_1}\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}(x/\sigma_1)^2}dx = \frac{1}{\sqrt{2\pi}\,\sigma_1}\left[\frac{\sqrt{\pi}}{2\sqrt{\left(\frac{1}{2\sigma_1^2}\right)^3}}\right]$$
$$= \sigma_1^2$$

$$\mathbb{E}\{(\xi_1 - \mathbb{E}\xi_1)(\xi_2 - \mathbb{E}\xi_2)\} = \mathbb{E}[\xi_1\xi_2 - \mu_1\xi_2 - \mu_2\xi_1 + \mu_1\mu_2]$$
$$= \iint_{-\infty}^{\infty} x_1x_2 P_{\xi_1\xi_2}(x_1,x_2)dx_1dx_2 + \int_{-\infty}^{\infty}\mu_1 x_2 P_{\xi_2}(x_2)dx_2$$
$$+ \int_{-\infty}^{\infty}\mu_2 x_1 P_{\xi_1}(x_1)dx_1 + \mu_1\mu_2$$

SySc 512 Session 18-19 Optimal Decoding
[Dayan & Abbott] Theo Neuro (2001)
Context: Sensor array → spiking neurons
⇒ rate vector $\vec{r}$

obj

*→ What is the probability that the true stim is in $[s, s \pm \Delta s]$ given $\vec{r}$: $P\{s \pm \Delta s \mid \vec{r}\}$?

Bayesian Inference:

$$P(s \mid \vec{r}) = \frac{P(\vec{r} \mid s)\, p(s)}{p(\vec{r})}$$

Estimate stim: ~~position~~ $\hat{s}_B$ given $\vec{r}$ via Bayes.
Let $L(s, \hat{s}_B)$ be cost of reporting $\hat{s}_B$ if really s.

$$\Rightarrow \min_{\hat{s}_B} \int L(s, \hat{s}_B)\, p(s \mid \vec{r})\, ds$$

If $L = (s - \hat{s}_B)^2 \Rightarrow$ min is just mean:

$$\hat{s}_B = \int s\, p(s \mid \vec{r})\, ds$$

But dependent on "~~cost~~-function" form
eg. If $L = |s - \hat{S}_B| \Rightarrow \hat{S}_B$ is median.

Maximum a posteriori (MAP)
Find $\hat{s}_M$ that max $p(\hat{s}_M \mid \vec{r})$
⇒ most likely stim for $\vec{r}$.
If "prior" $p(s)$ is indep of s (uniform distr)
⇒ $p(s \mid \vec{r})$ & $p(\vec{r} \mid s)$ have same s-depend.
(via Bayes rule)

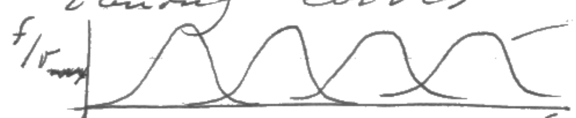
⇒ Maximum Likelihood (ML) estim: $\hat{s}_{ML}$

$$\max_{\hat{s}_{ML}} p(\vec{r} \mid s_{ML})$$

+(special case of MAP)

SySc 512 Optimal Decoding (cont)

Examp Array of N neurons w/ Normal Distr tuning curves



$$\Rightarrow f_a(s) = r_m \exp\left[-\frac{1}{2}\left(\frac{s-s_a}{\sigma_a}\right)^2\right] \quad (s_a \equiv \text{prefered value})$$

Model of sensory system

Assume 1: $\Sigma_a f_a(s)$ is indep of s

Assume 2: Spk rate of neuron-a, $r_a = n_a/T$

$$= \frac{\#\text{ spikes}}{\text{trial duration}}$$

Assume 3: Variance of Poisson process,

$\mathbb{E}\, r_a = f_a(s)$

$\Rightarrow$ Prob of stim s, evoking $n_a = r_a T$ spks is

$$P(r_a \mid s) = \frac{(f_a(s)T)^{r_a T}}{(r_a T)!} e^{-f_a(s)T}$$

$$\therefore P(\vec{r} \mid s) = \prod_{a=1}^{N} P(r_a \mid s)$$

Max Likelihood: isolate s-dep:

$$\ln P(\vec{r} \mid s) = T\, \Sigma_a r_a \ln(f_a(s)) + \underbrace{\cdots\cdots}_{}$$

{Terms indep of s like $\Sigma_a f_a(s)$ from expon}

$$\max_{\hat{s}_{ML}} \ln P(\vec{r} \mid \hat{s}_{ML}) \Rightarrow \max_{\hat{s}_{ML}} \left(\Sigma_a r_a \ln(f_a(s))\right)$$

1st order condition: $\displaystyle\sum_{a=1}^{N} r_a \frac{f_a'(\hat{s}_{ML})}{f_a(\hat{s}_{ML})} = 0$

Since $f_a'(s)/f_a(s) = (s_a - s)/\sigma_a^2$

$$\Rightarrow \hat{s}_{ML} = \frac{\Sigma r_a (s_a/\sigma_a^2)}{\Sigma s_a/\sigma_a^2} \xrightarrow{\sigma_a \to \sigma} \frac{\Sigma r_a s_a}{\Sigma r_a}$$

$\therefore$ Rates average over prefered values of encoding neurons.

SySc 512 Optimal Decoding (cont.)

MAP decoding

If $p(s)$ is not uniform, $\Rightarrow \hat{s}_M \neq \hat{s}_{ML}$

Compute $p(s|\vec{r})$ from Bayes rule:

$$\ln p(s|\vec{r}) = \ln P(\vec{r}|s) + \ln p(s) - \cancel{p(\vec{r})}$$

s-indep

$$= T \sum_{a=1}^{N} r_a \ln(f_a(s)) + \ln p(s) + \cdots$$

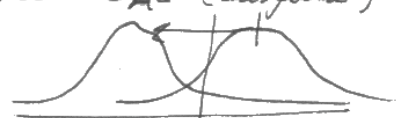
1st order condition

$$T \sum_a \frac{r_a f_a'(\hat{s}_M)}{f_a(\hat{s}_M)} + \frac{p'(\hat{s}_M)}{p(\hat{s}_M)} = 0$$

Assume: $p(s)$ is Normal dist'n: $\mathcal{N}(s_{prior}, \sigma_{prior})$

$$\Rightarrow \hat{S}_M = \frac{T \sum r_a \, s_a/\sigma_a^2 + s_p/\sigma_p^2}{T \sum r_a/\sigma_a^2 + 1/\sigma_p^2}$$

$\therefore$ If $s_p \neq 0 \Rightarrow$ shift max $\hat{s}_M$ from $\hat{s}_{ML}$ (uniform) because more likely near s_p



Accuracy of $\hat{s}$ characterized by

1) Bias: $b(s) = \mathbb{E}\hat{s} - s$ w/ $\mathbb{E}\hat{s}$ across trials; s is true value

2) Variance: $\hat{\sigma}^2(s) = \mathbb{D}\hat{s} = \mathbb{E}(\hat{s} - \mathbb{E}\hat{s})^2$

Thus, the estimation error is:

$$\mathbb{E}(\hat{s} - s)^2 = \mathbb{E}(\hat{s} - \mathbb{E}\hat{s} + \hat{b}(s))^2$$
$$= \hat{\sigma}^2(s) - \hat{b}^2(s)$$

Trade-off or minimize of each.

SySc 512 Optimal Decoding

Fisher Info: measure of encoding accuracy

Accuracy of any decoder to obtain $\hat{s}$ limited by Cramér-Rao bound

$$\hat{\sigma}^2(s) \geq \frac{(1-\hat{b}'(s))^2}{I_F(s)}, \quad \hat{b}'(s) = \frac{d}{ds}\hat{b}(s)$$

w/ $I_F(s) = \int \left(-\frac{\partial^2}{\partial s^2} \ln p(r|s)\right) p(r|s)\, dr$

- Assumed: $\vec{r}$ is continuous
- Prf by Cauchy Schwartz inequality: $\langle A^2 \rangle \langle B^2 \rangle \geq \langle AB \rangle^2$

I_F is expected curvature of log-likelihood @ s

Limit of decoding accuracy
$\Rightarrow$ limit of encoding accuracy

$N \to \infty \Rightarrow \hat{\sigma}^2_{ML}(s) = 1/I_F(s)$ (unbiased) $\Rightarrow$ "efficient"

Finite # neurons, discrete analogue:

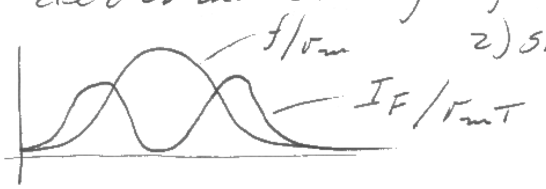
$$I_F(s) = T \sum_{a=1}^{N} (\mathbb{E} r_a) \left[\left(\frac{f_a'(s)}{f_a(s)} \right)^2 - \frac{f_a''(s)}{f_a(s)} \right]$$

$(\mathbb{E} r_a) = (f_a(s))$

Assume evenly spaced sensors. $\Rightarrow \sum f_a''/f_a = 0$

$$\Rightarrow I_F = T \sum_a \frac{(f_a'(s))^2}{f_a(s)}$$

I_F determined by 1) # of active neurons 2) slope of tuning curve



Trade-off wide $\Rightarrow$ many active vs. narrow $\Rightarrow$ steep slope

$$I_F(s) = T \sum_a \frac{r_m (s-s_a)^2}{\sigma^4} \exp\left[-\frac{1}{2}\left(\frac{s-s_a}{\sigma}\right)^2\right], \quad \sigma_a = \sigma$$

(sensor density factor)

$$\approx \rho_s T \int_{-\infty}^{\infty} \frac{r_m (s-s')^2}{\sigma^4} \exp\left[-\frac{1}{2}\left(\frac{s-s'}{\sigma}\right)^2\right] ds'$$

$$= \frac{1}{\sigma} \sqrt{2\pi} (\rho_s \sigma) r_m T \qquad (\rho_s \sigma \approx \text{\# active neurons})$$

$\propto \frac{1}{\sigma} \Rightarrow I_F$ increases w/ narrow tuning

$\boxed{\text{(Discriminability: } d = \Delta s \sqrt{I_F(s)})}$