SySc512 Random Variables & Distributions (14)

Given a sample space $\Omega$, w/ events $\omega \in \Omega$

· "Random variables" map events to numbers.

$$\xi = \xi(\omega) \in \mathbb{R}$$

$P\{x' \le \xi \le x''\}$ is probability of events $\{x' \le \xi \le x''\}$

"Probability distribution" expresses $P\{\ \}$ over an interval

$$P_\xi(x) = P\{\xi = x\}, \quad x \in [x', x'']$$

Discrete rand. variables:

Examp 1) coin toss: heads/tails $\longmapsto \xi(h) = 1,\ \xi(t) = 0$

$$P_\xi(0) = \tfrac{1}{2},\ P_\xi(1) = \tfrac{1}{2} \Rightarrow \sum_x P_\xi(x) = 1$$

Examp 2) 2 dice: $\xi(\text{roll}) = \#$ spots

$$\sum_x P_\xi(x) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \cdots = \underline{1}$$

| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| | (2,1) | (3,1) | (4,1) | (5,1) | (6,1) |
| | | (2,2) | (2,3) | (2,4) | (2,5) |
| | | | (3,2) | (4,2) | (5,2) |
| | | | | (3,3) | (4,3) |
| | | | | | (3,4) |

Sum over all $x = \xi(\omega) \Rightarrow$ sum over all $\omega \in \Omega$

$$\sum_x P_\xi(x) = \underline{1}$$

$\boxed{\text{Matlab Demo 1}}$ (distrib1a.m, distrib1b.m)

Continuous rand variables:

Example: position of coin tossed

"Probability density" ~~[illegible]~~

Since, $P\{\xi = x\} = 0$ for continuous variables

$$\Rightarrow P\{x \le \xi \le x + dx\} = p_\xi(x)\,dx$$

Integral over all $x = \xi(\omega)$: $\int_x p_\xi(x)\,dx = 1$

Cumulative Distribution Function:

$$\Phi_\xi(x) = P\{\xi \le x\}$$

$$\Rightarrow \Phi_\xi(x) = \sum_{-\infty}^{x} P_\xi(x) \quad \text{or} \quad \Phi_\xi(x) = \int_{-\infty}^{x} p_\xi(x)\,dx$$

on an interval:

$$P\{x_o \le \xi \le x_f\} = \Phi(x_f) - \Phi(x_o)$$

SySc 512 Session 14, Rand Variabls & Distr (cont.)

Joint probability distrib: ($x \to \vec{x}$)

$$P_{\xi_1, \xi_2}(x_1, x_2) = P\{\xi_1 = x_1, \xi_2 = x_2\}$$

Discrete:

$$P\{(\xi_1, \xi_2) \in B\} = \sum_{x_1, x_2} P_{\xi_1, \xi_2}(x_1, x_2)$$

Continuous:

$$= \iint_B P_{\xi_1 \xi_2}(x_1, x_2)\, dx_1\, dx_2$$

If indep variables:

$$\Rightarrow P_{\xi_1 \xi_2}(x_1, x_2) = P_{\xi_1}(x_1) P_{\xi_2}(x_2)$$

$\boxed{\text{Matlab Demo 2}}$ (distr2a.m, distr2b.m)

Uniform Distrib (w/in [a, b])

$$P\{x' \le \xi \le x''\} = \frac{x'' - x'}{b - a} = \int_{x'}^{x''} \frac{dx}{b-a} = 1$$

Density for uniform distr:

$$P_\xi(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$ (prob in x-interval)

$$\Phi_\xi(x) = \begin{cases} \frac{x}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x < a \\ 1 & \text{if } x > b \end{cases}$$

Expectation

Discrete: $\mathbb{E}\xi = \sum_{x=-\infty}^{\infty} x P_\xi(x)$ (mean value or average)

Let $\eta$ be a function of $\xi \ni \eta = \eta(\xi)$

$$\Rightarrow \mathbb{E}\eta = \mathbb{E}\eta(\xi) = \sum_{x=-\infty} \eta(x) P_\xi(x)$$

Properties:

a) $\mathbb{E}1 = 1$

b) $\mathbb{E}(a\xi_1 + b\xi_2) = a\mathbb{E}\xi_1 + b\mathbb{E}\xi_2$

c) $|\mathbb{E}\xi| \le \mathbb{E}|\xi|$ (triangle inequality)

d) If $\xi_1 \le \xi_2 \Rightarrow \mathbb{E}\xi_1 \le \mathbb{E}\xi_2$

c) If $\xi_1$ & $\xi_2$ independent $\Rightarrow \mathbb{E}(\xi_1 \xi_2) = \mathbb{E}\xi_1 \mathbb{E}\xi_2$

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Continuous: $\mathbb{E}\xi = \int_{-\infty}^{\infty} x\, p_\xi(x)\,dx$

(by continuum limit of discrete & Cauchy convergence)

Likewise, $\mathbb{E}\eta(\xi) = \int_{-\infty}^{\infty} \eta(x)\, p_\xi(x)\,dx$

For joint probability densities: $\xi, x \to \vec{\xi}, \vec{x}$

Examp: Uniform distrib:

$$\mathbb{E}\xi = \int_a^b x\left(\frac{1}{b-a}\right)dx = \frac{1}{2}\left(\frac{b^2-a^2}{b-a}\right) = \frac{1}{2}(a+b)$$

Variance (dispersion, deviation from mean)

Let $\mu = \mathbb{E}\xi$

$$\sigma_\xi^2 = \mathbb{D}\xi = \mathbb{E}(\xi-\mu)^2$$
$$= \mathbb{E}\xi^2 - 2\mu\mathbb{E}\xi + \mu^2 = \mathbb{E}\xi^2 - 2\mu^2 + \mu^2$$
$$= \mathbb{E}\xi^2 - (\mathbb{E}\xi)^2$$

Properties:

1) $\mathbb{D}1 = 0$

2) $\mathbb{D}(a\xi) = a^2\mathbb{D}\xi$

3) $\mathbb{D}(\xi_1+\xi_2) = \mathbb{E}(\xi_1-\mu_1+\xi_2-\mu_2)^2$

$$= \mathbb{E}(\xi_1-\mu_1)^2 + \cancel{2\mathbb{E}\{(\xi_1-\mu_1)(\xi_2-\mu_2)\}}^{=0 \text{ if indep}} + \mathbb{E}(\xi_2-\mu_2)^2$$
$$= \mathbb{D}\xi_1 + \mathbb{D}\xi_2$$

Non-independent => correlation coeff:

$$r = \frac{\mathbb{E}\{(\xi_1-\mu_1)(\xi_2-\mu_2)\}}{\sigma_1\sigma_2}$$

($r=0$ for indep variables)

$\boxed{\text{Matlab Demo 3}}$

SySc 512, Session 14, Distrs (cont)

Bernoulli trials: $p = P(\text{count})$

Let $q = 1 - p$, prob. of failure

Let $\omega$ = sequence during $n$ trials: $\underbrace{1011\cdots0010}_{n\text{-times}}$

Let $\xi(\omega) = k$ : # of successes

If $p = q$ ⇒ coin flip. | If $n=3$, $k=2$, ⇒ $(110)(101)(011)$ ⇒ $n$ taken $k$ @ a time

$$\Rightarrow P_\xi(k) = C^n_k\, p^k q^{n-k}, \quad k = 0, 1, \ldots, n$$

w/ $C^n_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Since $\xi = \sum_{k=1}^n \xi_k$ (sum of rand variables)

and $\mathbb{E}\xi_k = p \Rightarrow \mathbb{E}\xi = np$

$$\mathbb{D}\xi_k = \mathbb{E}\xi_k^2 - (\mathbb{E}\xi_k)^2$$

$$= p - p^2 = p(1-p) = pq \Rightarrow \mathbb{D}\xi = pq$$

(because $\xi_k^2 = 1^2 = 1$

Let $p \to 0$ as $n \to \infty$ such that $\boxed{a = np}$ is const

$$\lim_{n\to\infty} P_\xi(k) = \lim \frac{n!}{k!(n-1)!}\left(\frac{a}{n}\right)^k\left(1 - \frac{a}{n}\right)^{n-k}$$

$$= \lim \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{a^k}{n^k}\right)\left(1-\frac{a}{n}\right)^{-k}\left(1-\frac{a}{n}\right)^n$$

$$= \lim \frac{n(n-1)\cdots(n-k+1)}{n^k}\left(\frac{a^k}{k!}\right)\left(1-\frac{a}{n}\right)^{-k}\left(1-\frac{a}{n}\right)^n$$

$$= 1 \cdot \frac{a^k}{k!} \cdot 1 \cdot e^{-a}$$

⇒

$$P_\xi(k) \to \frac{(np)^k}{k!} e^{-np}$$ Poisson distrib

SySc 512 ~~Lesson~~ Lesson 14 Distr.s (cont.)

Poisson Distrib: $P_\xi(k) = \frac{(np)^k}{k!} e^{-np}$ [k successes, n events, p - prob of success]

$$\mathbb{E}\xi = \sum_{k=0}^{\infty} k\, P_\xi(k)$$
$$= \sum_{k=0}^{\infty} k \frac{a^k}{k!} e^{-a} = a e^{-a} \sum_{k=1}^{\infty} \frac{a^{k-1}}{(k-1)!} = a e^{-a} e^{a} = a$$

$\mathbb{D}\xi$, solve by $\lim_{n\to\infty} \underbrace{\mathbb{D}\xi(\text{Bernoulli})}_{npq} = \lim_{n=\infty} a(1-\frac{a}{n}) \to a$

Limit $n \to \infty$, could be thought of as grid getting finer $\to$ continuum limit

$\boxed{\text{Matlab Demo 4}}$ (distrib4a.m, distrib4b.m)

Exponential Distrib:

Survival function: $S(t) = \begin{cases} a e^{-dt} & t > 0 \\ 0 & \text{otherwise} \end{cases}$

$\underbrace{\mathbb{E}\xi = \frac{1}{d}}_{\text{lifetime}}$ & $\mathbb{D}\xi = \frac{1}{d^2}$

lifetime, half-life $= \frac{1}{d} \ln(2)$

Normal (Gaussian) Distrib

$$P_G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad \text{w/} \quad \begin{matrix} \mu = \mathbb{E}\xi \\ \sigma^2 = \mathbb{D}\xi \end{matrix}$$

Or, in $\vec{x} \in \mathbb{R}^n$

$$P_G(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |r|}} \exp\left\{-\tfrac{1}{2}(\vec{x}-\vec{\mu})\, \mathbb{R}(\vec{x}-\vec{\mu})\right\}$$

w/ $\vec{\mu} = \mathbb{E}\vec{\xi}$, $\mathbb{R} = \mathbb{E}(\vec{\xi}-\vec{\mu})(\vec{\xi}-\vec{\mu})^T \equiv$ cov. matrix

(if indep $\Rightarrow \mathbb{R} = \mathbb{1}$)

$\boxed{\text{Matlab Demo 5}}$