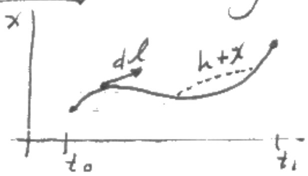
SySc 512, Session 12, Variational Calculus

Optimize objective function over space of Functions

$f(x) \in \mathcal{F}$, feasible set of functions (instead of $x \in X$)

Functional $\Phi : \mathcal{F} \to \mathbb{R}$

Examp: Length of curve, $\gamma = \{(x,t) : x = x(t),\ t_0 < t < t_1\}$



$$l(t) = \int_{t_0}^{t_1} \left(\frac{dl}{dt}\right) dt \quad \text{(for each path)}$$

$$\frac{dl}{dt} = \sqrt{(dt/dt)^2 + (dx/dt)^2}$$

$$\Rightarrow \Phi(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(x(t), \dot{x}(t), t)\, dt$$

Minimize w/ 1st order condition: $\delta\Phi = 0$

1st Variation (functional derivative)

$$\Phi(\gamma + h) - \Phi(\gamma) = \delta\Phi + o(h^2)$$

$\Rightarrow$

$$\delta\Phi = \int_{t_0}^{t_1} [\mathcal{L}(x+h, \dot{x}+\dot{h}, t) - \mathcal{L}(x, \dot{x}, t)]\, dt$$

Expand 1st term about $h(t) = 0$

$$= \int_{t_0}^{t} \left[\frac{\partial}{\partial x}\mathcal{L} \cdot h + \frac{\partial}{\partial \dot{x}}\mathcal{L}\, \dot{h}\right] dt + o(h^2)$$

$\Rightarrow$

$$\delta\Phi(h, \gamma) = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial x} h + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{h}\right] dt$$

Integrating by parts: $\int u\, dv = uv - \int v\, du$

w/ $u = \partial\mathcal{L}/\partial\dot{x}$, $v = h$

$$\Rightarrow \int_{t_0}^{t_1} \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{h}\, dt = \cancel{\left[\frac{\partial \mathcal{L}}{\partial \dot{x}} h\right]_{t_0}^{t_1}}^{\to 0} - \int_{t_0}^{t_1} \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] h\, dt$$

$$\Rightarrow \delta\Phi(h, \gamma) = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] h\, dt$$

$\delta\Phi(h, \gamma) = 0$ for all h iff $\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0}$ ← Euler-Lagrange Eq.

Examp: Length of curve: $\mathcal{L}(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$$

Euler-Lagrang eq $\Rightarrow \partial\mathcal{L}/\partial\dot{x} = 0 \Rightarrow \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = \text{const.}$

$$\Rightarrow \dot{x}^2 = c(1 + \dot{x}^2) \Rightarrow \dot{x} = \sqrt{\frac{c}{1-c}}$$

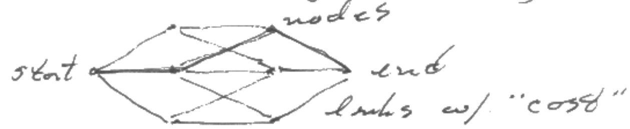
$$\therefore x(t) = t_1 + \sqrt{\frac{c}{1-c}}\, t \quad \Rightarrow \text{straight line.}$$

SySc 512, Session 12 Dynamic Programming

Types of problems:

1) Minimum Chain

=> find least "cost"



2) Optimal Control:

- Sequence of "controllable" states: $s_1, s_2, \ldots s_N$
- State transitions: $s_{n+1} = g(n, s_n, u_n)$
- Sequence of controls (decisions): $u_1, u_2, \ldots$

$$\boxed{s_1} \to \underset{\downarrow r_1}{\overset{u_1 \downarrow}{g(1, s_1, u_1)}} \to \boxed{s_2} \to \underset{\downarrow r_2}{\overset{u_2 \downarrow}{g(2, s_2, u_2)}} \to \cdots \to \underset{\downarrow r_{N-1}}{\overset{u_{N-1} \downarrow}{g(N-1, s_{N-1}, u_{N-1})}} \to \boxed{s_N}$$

- Return value (cost, reward), $r_n = r(n, s_n, u_n)$

3) Resource Allocation

- Resources: u_n, Costs: r_n

Optimum Problem: Find optimal policy: $\{u_1, u_2, \ldots, u_N\}$ to minimize total cost (max reward):

$$\min_{\{u_n\}} \sum_n r_n$$

subject to $s_{n+1} = g(n, s_n, u_n)$

Bellman's Princ. of Optimality

If an optimal policy,

=> for any u_n, remaining decisions must be optimal.

Value function (sum of future costs)

$$V(n, s_n) = \min_{\{u_n\}} \sum_{m=n}^{N} r_m \quad , \text{ s.t. } s_{n+1} = g(n, s_n, u_n)$$

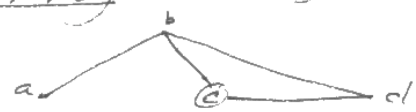
=> Bellman's equation:

$$V(n, s_n) = \min_{\{u_n\}} \left[r_n + V(n+1, s_{n+1}) \right]$$

∴ Min each step & whole thing is minimized.

SySc 512 Session 12 Dyn Prog (cont.)

Prf: Bellman's Principle



Assertion:

If abd is optimal path $a \to d$

$\Rightarrow$ bd is optimal.

Suppose $\exists$ c such that bcd is optimal path $b \to d$

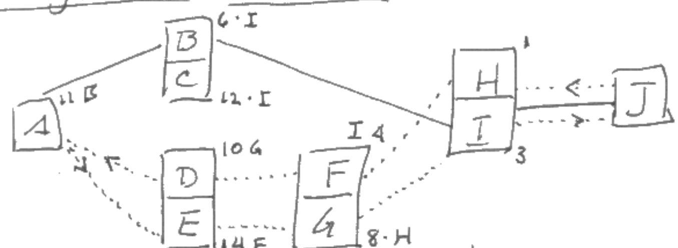
$\Rightarrow V(bd) > V(bcd)$

But since $V(abd) = V(ab) + V(bd) > V(ab) + V(bcd)$

$\Rightarrow V(abd) > V(abcd)$

$\times$ because $V(abd)$ is optimal

Stagecoach Problem



	B	C	D	E
A	5	2	4	1

	H	I
B	8	3
C	12	9

	F	G
D	7	2
E	10	8

	H	I
F	6	1
G	7	6

	J
H	1
I	3

Greedy path: A E G I J , 18

Rev Greedy " : J H E D A , 18

Optimal: A B I J , 11