SySc 512 Homework 3 Solutions

1) $A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0.8 & 1.8 \\ 0 & 1.25 \end{bmatrix}$

a) $Tr(A_1) = 5$ $Tr(A_2) = 0$ $Tr(A_3) = 1.4$, $Tr(A_4) = 2.05$

$Det(A_1) = 4$ $det(A_2) = -1$ $det(A_3) = 2(0.7)^2 = 0.98$ $Tr(A_4) = 0.8(1.25) = 1$

Recall Eigenvalues:

Since A_1 is diagonal $\Rightarrow \lambda = 4, 1$

A_2 is diagonal $\Rightarrow \lambda = 1, -1$

A_3: $(0.7 - \lambda)^2 + 0.7^2 = 0$

$\lambda^2 - 2 \cdot 0.7\lambda + (0.7)^2 + 0.7^2 = 0$

$\lambda^2 - 2 \cdot 0.7\lambda + 2 \cdot 0.7^2 = 0$

$\lambda = \frac{+2 \cdot 0.7 \pm \sqrt{4(0.7)^2 - 4 \cdot 2 \cdot 0.7^2}}{2} = 0.7 \pm i\,0.7$

Real Canonical form

A_4: $(0.8 - \lambda)(1.25 - \lambda) - 0 = 0$

$(\frac{4}{5} - \lambda)(\frac{5}{4} - \lambda) - 0 = 0$ $\Rightarrow \boxed{\lambda = \frac{4}{5}, \frac{5}{4}}$

$\lambda^2 - (\frac{5}{4} + \frac{4}{5})\lambda + 1 = 0$

$\lambda^2 - 2.05\lambda + 1 = 0$

$\lambda = \frac{2.05 \pm \sqrt{(2.05)^2 - 4}}{2} = \frac{2.05 \pm 0.45}{2} = \begin{matrix} 0.8 \\ 1.25 \end{matrix}$

Eigenvalues of triangular matrix are diag. Det is product of diag elements

Eigenvectors:

A_1: $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = 4\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \Rightarrow \begin{bmatrix} 4V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 4V_{11} \\ 4V_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 1\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 4V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

A_3: $\begin{bmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = 0.7(1+i)\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \Rightarrow \begin{bmatrix} V_{11} - V_{21} \\ V_{11} + V_{21} \end{bmatrix} = \begin{bmatrix} (1+i)V_{11} \\ (1+i)V_{21} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -V_{21} \\ V_{11} \end{bmatrix} = i\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \Rightarrow \begin{cases} iV_{11} = -V_{21} \\ V_{11} = iV_{21} \end{cases} \Rightarrow \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\begin{bmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 0.7(1-i)\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} V_{12} - V_{22} \\ V_{12} + V_{22} \end{bmatrix} = \begin{bmatrix} (1-i)V_{12} \\ (1-i)V_{22} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -V_{22} \\ V_{12} \end{bmatrix} = -i\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \Rightarrow \begin{cases} iV_{22} = iV_{12} \\ V_{12} = -iV_{22} \end{cases} \Rightarrow \begin{bmatrix} 1 \\ i \end{bmatrix}$

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A_4: $\begin{bmatrix} 0.8 & 1.8 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0.8\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} 0.8V_{11} + 1.8V_{12} \\ 1.25\,V_{12} \end{bmatrix} = 0.8\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$

$\Rightarrow 0.8V_{11} + 1.8V_{12} = 0.8V_{11}$, $1.25\,V_{12} = 0.8V_{12}$ $\Rightarrow V_{11} = ?$, $V_{12} = 0$ $\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0.8 & 1.8 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 1.25\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix}$

$\Rightarrow 0.8V_{21} + 1.8V_{22} = 1.25V_{21} \Rightarrow 1.8V_{22} = .45V_{21}$

$1.25V_{22} = 1.25V_{22} \Rightarrow \begin{bmatrix} 1.8 \\ .45 \end{bmatrix}$

b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0.9701 \\ 0 & 0.2425 \end{bmatrix} \Rightarrow Q^{-1} = \frac{1}{0.2425}\begin{bmatrix} 0.2425 & -0.9701 \\ 0 & 1 \end{bmatrix}$

$Q\,D\,Q^{-1} = \begin{bmatrix} 1 & 0.9701 \\ 0 & 0.2425 \end{bmatrix}\begin{bmatrix} 0.8 & 0 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} 1 & -\frac{0.9701}{0.2425} \\ 0 & \frac{1}{0.2425} \end{bmatrix}$

$= A_4$

2) Matlab

Complex eigenvectors:

We had solution $X(t) = \vec{V}_1 e^{\lambda_1 t} + \vec{V}_2 e^{\lambda_2 t}$

$= \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{0.7t}e^{i0.7t} + \begin{bmatrix} 1 \\ i \end{bmatrix} e^{0.7t}e^{-i0.7t}$

$= e^{0.7t}\begin{bmatrix} e^{i0.7t} + e^{-i0.7t} \\ -ie^{i0.7t} + ie^{-i0.7t} \end{bmatrix}$

Since, $2\cos\theta = e^{i\theta} + e^{i\theta}$

$2i\sin\theta = e^{i\theta} - e^{i\theta}$

$\rightarrow$

$= e^{0.7t}\begin{bmatrix} 2\cos(0.7t) \\ -i(2i)\sin(0.7t) \end{bmatrix} = 2e^{0.7t}\begin{bmatrix} \cos(0.7t) \\ \sin(0.7t) \end{bmatrix}$