SySc 512 – Quantitative Methods of Systems Science

Homework 6: Uncertainty.

(1) A point process describes a series of events where the probability of an event at any time is constant. The random variable, τ, quantifies the time of an event.

(a) The inter-event density function is an exponential function:

$$P_\tau(t) = Re^{-Rt}, \text{ where } t > 0.$$

Compute the expectation, $\mathbf{E}\tau$, and variance, $\mathbf{D}\tau$ of the time until the first event.
Graph $P_\tau(t)$ as a function of t and mark the mean and variance on the graph.

(b) The probability density function for the time between every n^{th} event is gamma distribution density:

$$P^n_\tau(t) = \frac{(nR)^n}{(n-1)!} t^{n-1} e^{-nRt},$$

where $t > 0$, and n is the number of events.
Compute the expectation, $\mathbf{E}\tau$, and variance, $\mathbf{D}\tau$ of the time until the third event.
Graph $P^3_\tau(t)$ as a function of t and mark the mean and variance on the graph.

Hint: you will need the integral:

$$\int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$$

(2) The joint probability density of two random variables, ξ_1 and ξ_2, is given by the bivariate Gaussian density function:

$$P_{\xi_1,\xi_2}(\hat{x}_1, \hat{x}_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\hat{x}_1^2 - 2\rho\hat{x}_1\hat{x}_2 + \hat{x}_2^2}{2(1-\rho^2)}\right],$$

where $\hat{x}_i$ is the normalized variable, $\hat{x}_i = (x - \mathbf{E}\xi_\mathbf{i})/\sigma_\mathbf{i}$, $\sigma_i = \mathbf{D}\xi_\mathbf{i}$, and $\rho < 1$.
Compute the *marginal* density function for ξ_1, ie. $P_{\xi_1}(\hat{x}_1)$.
Graph $P_{\xi_1,\xi_2}(\hat{x}_1, \hat{x}_2)$ and $P_{\xi_1}(\hat{x}_1)$ for $\rho = 0$, 0.5, and 0.9.
Using the marginal and the joint density function, compute $\mathbf{E}\xi_\mathbf{1}$, $\mathbf{D}\xi_\mathbf{1}$, and $\mathbf{E}[(\xi_\mathbf{1} - \mathbf{E}\xi_\mathbf{1})(\xi_\mathbf{2} - \mathbf{E}\xi_\mathbf{2})]$.