SysSc 512 – Quantitative Methods of Systems Science

PRACTICE FINAL

(1) A gradient dynamical system has the following potential function:

$$V(x,y) = \frac{1}{2}(x+1)^2 + y^2$$

(a) Write the differential equations that describe the dynamical system.
(b) Use the differential equations to find the fixed point of the gradient dynamical system.
(c) Compute the Jacobian of the dynamical system at the fixed point.
(d) Compute the eigenvalues the Jacobian at the fixed point.
(e) Use the first order conditions to find an extremum of V(x,y).
(f) Use the second order conditions to determine if the extremum is a minimum.
(g) Compute the vector of steepest descent of $V(x,y)$ at the point $(x,y) = (0,1)$.

(2) Consider the following objective function:

$$f(x,y) = \frac{1}{2}x^2 + (x-1)y + \frac{1}{2}y^2$$

subject to the constraint:

$$c(x,y) = x - y - 1 = 0$$

(a) Does the unconstrained objective functions have an extremum? What is it?
(b) Write the Lagrangian, $\mathcal{L}(x,y,\lambda)$.
(c) Use the first order conditions to find the extremum of $V(x,y)$ subject to the constraint.
(d) Use the second order conditions to determine if the extremum is a minimum:
  (i) Compute the Hessian of the objective function, $H(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix}$.
  (ii) Compute the gradient of the constraint, $\nabla c(x,y)$.
  (iii) Find a vector, $\vec{n}$, in the null space of $\nabla c(x,y)$, so that $\vec{n} \cdot \nabla c(x,y) = 0$.
  (iv) Project the Hessian onto the null space of $\nabla c(x,y)$.
  (v) Determine if the projected Hessian is positive definite or negative definite. Is the extremum a minimum?

(3) The joint probability density of two random variables, ξ_1 and ξ_2, is:

$$P_{\xi_1,\xi_2}(x_1,x_2) = \frac{\sqrt{2}}{4\pi} \exp\left[-(\frac{x_1^2}{2} - \frac{x_1 x_2}{\sqrt{2}} + x_2^2)\right],$$

(a) Compute the marginal density function for ξ_1, ie. $P_{\xi_1}(x_1)$.
Hint: you will need the integral:

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{1 \cdot 2 \cdot 3 \cdots (n+1)\sqrt{\pi}}{\sqrt{2^n a^{n+1}}}, \text{ for } n = 0, 2, 4, \ldots.$$

(b) What is the mean and the variance of the random variable, ξ_1?
(c) Compute the conditional probability, $P_{\xi_1,\xi_2}(x_2|x_1)$.